MATH 40 LECTURE 10: BASES, DIMENSION AND RANK

DAGAN KARP

In this lecture we return to our discussion of subspaces. We learn what a basis is, and use it to define the dimension of a subspace. We also revisit the notion of rank, and obtain a second part of the Fundamental Theorem of Line Integrals.

Definition 1. *Let S be a subspace of $\mathbb{R}^n$. A subset $B \subset S$ of vectors in S is called a* basis *of S if and only if*

(1) B *spans* S *and*
(2) B *is linearly independent.*

Example 2. *Consider the following.*

(1) *The vectors $\vec{e}_1, \ldots, \vec{e}_n$ form a basis of $\mathbb{R}^n$.*
(2) *The vectors $(1, 0)$ and $(1, 1)$ are also a basis of $\mathbb{R}^2$.*

Definition 3. *The vectors $\{e_1, \ldots, e_n\}$ are called the* standard basis *of $\mathbb{R}^n$.*

Example 4. *Find a basis for $\operatorname{span}\{(1, 2), (2, 7), (-3, -6)\}$. Note that $\{(1, 2), (2, 7)\}$ is a linearly independent set, and $(, 3, 6) \in \operatorname{span}\{(1, 2), (2, 7)\}$. Thus $\{(1, 2), (2, 7)\}$ is a basis of*

$$\operatorname{span}\{(1, 2), (2, 7), (-3, -6)\}.$$

Remark 5. We will study a matrix A by searching for bases of the row space, column space and null space of A. How can we do so in practice? Let R be the reduced row echelon form of A. Then the nonzero row vectors of R form a basis of $\operatorname{row}(A)$. Also, the leading columns form a basis of $\operatorname{col}(A)$. We use the free variables of $R\vec{x} = \vec{0}$ to determine a basis of $\operatorname{null}(A)$.

Theorem 6 (Basis Theorem)**.** *Let S be a subspace of $\mathbb{R}^n$. Then any two bases of S have the same number of vectors.*

Definition 7. *Let S be a subspace of $\mathbb{R}^n$. The* dimension *of S is the number of elements in any basis of S.*

Example 8.

$$\boxed{\dim(\mathbb{R}^n) = n.}$$

Theorem 9. *For any matrix A,*

$$\dim \operatorname{row}(A) = \dim \operatorname{col}(A).$$

Corollary 10. *The rank of A is equal to the dimension of its row space, which is the same as the dimension of the column space of A.*

Date: February 7, 2012.

These are lecture notes for HMC Math 40: Introduction to Linear Algebra and roughly follow our course text *Linear Algebra* by David Poole.

Example 11. *Consider*

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 7 & -6 \end{pmatrix}.$$

We saw above that $\dim \operatorname{col}(A) = 2$*, thus* $\operatorname{rank}(A) = 2$*. Note that we can also determine this from the rows of* A*. The reduced echelon form of* A *is*

$$R = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus R *has two nonzero rows. Hence, again,* $\operatorname{rank}(A) = 2$*.*

Definition 12. *The* nullity *of* A *is the dimension of the null space of A,*

$$\operatorname{nullity}(A) = \dim \operatorname{null}(A).$$

Theorem 13 (Rank-Nullity Theorem)**.** *If* A *is an* $m \times n$ *matrix, then*

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = n.$$

PROOF. Let R be the reduced row echelon form of A. Suppose $\operatorname{rank}(R) = r$. Then R has r leading 1's. Thus, by the Rank Theorem, there are $n-r$ free variables in the homogeneous system $(A|\vec{0})$. Each free variable has a corresponding basis element in the null space of A. Thus $\dim \operatorname{null}(A) = n - r$. Therefore

$$\operatorname{rank}(A) + \operatorname{nullity}(A) = r + (n - r) = n. \quad \square$$

Theorem 14 (Fundamental Theorem of Invertible Matrices Part II)**.** *Let* A *be an* $n \times n$ *matrix. The following are equivalent.*

- i A *is invertible.*
- ii $\operatorname{rank}(A) = n$.
- iii $\operatorname{nullity}(A) = 0$.
- iv *The columns of* A *are linearly independent.*
- v *The columns of* A *span* $\mathbb{R}^n$*.*
- vi *The columns of* A *are a basis of* $\mathbb{R}^n$*.*
- vii *The rows of* A *are linearly independent.*
- viii *The rows of* A *span* $\mathbb{R}^n$*.*
- ix *The rows of* A *form a basis of* $\mathbb{R}^n$*.*

PROOF.

$(i) \Rightarrow (ii)$ By FTIM I, A is invertible if and only if its reduced row echelon form is I_n. But I_n has n nonzero rows. Thus $\operatorname{rank}(A) = n$.

$(ii) \Leftrightarrow (iii)$ This holds by the Rank-Nullity Theorem.

$(ii) \Rightarrow (iv)$ If $\operatorname{rank}(A) = n$, then the linear system $A\vec{x} = \vec{0}$ has no free variables, and hence has only the trivial solution. Therefore the columns of A are linearly independent. Note that, by FTIM I, this also shows that A is invertible.

$(iv) \Rightarrow (v)$ If the columns of A are linearly independent, then $A\vec{x} = \vec{0}$ has only the trivial solution. Thus, by FTIM I, $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbb{R}^n$. Therefore every vector $\vec{b}$ can be written as a linear combination of the columns of A.

$(v) \Rightarrow (vi)$ Suppose the columns of A span $\mathbb{R}^n$. Then $\dim \operatorname{col}(A) = n$. But $\operatorname{rank}(a) = \dim \operatorname{col}(A)$, so $\operatorname{rank}(A) = n$. This is (ii). But we already showed $(ii) \Rightarrow (iv)$. Thus, the columns of A are linearly independent. They also span $\mathbb{R}^n$ by assumption. Therefore they are a basis of $\mathbb{R}^n$.

$(vi) \Rightarrow (ii)$

We have now proved $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (ii) \Rightarrow (i)$. Hence these statements are equivalent. Now, we claim that $\operatorname{rank}(A) = \operatorname{rank}(A^T)$ for any matrix A. Indeed,

$$\begin{aligned} \operatorname{rank}(A^T) &= \dim \operatorname{col}(A^T) \\ &= \dim \operatorname{row}(A) \\ &= \operatorname{rank}(A). \end{aligned}$$

Also, we know that A is invertible if and only if A^T is invertible.

Therefore, we may apply our proven results to A^T. But the columns of A^T are the rows of A. This completes the proof. □