MATH 40 LECTURE 11: LINEAR TRANSFORMATIONS

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In this lecture, we provide one interpretation of a matrix, giving the notion of matrix greater depth than just an array of numbers.

Definition 1. *A* linear transformation *from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$ *is a map* $T : \mathbb{R}^n \to \mathbb{R}^m$ *such that*

(1) $T(c\vec{v}) = cT(\vec{v})$*; and*
(2) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$*, for any vectors* $\vec{u}, \vec{v} \in \mathbb{R}^n$ *and scalar* c*.*

Remark 2. T is a linear transformation if and only if

$$T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v}),$$

for any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ and any scalars a and b.

Example 3. *Let* $T : \mathbb{R}^n \to \mathbb{R}^n$ *be given by*

$$T(\vec{v}) = 5\vec{v}.$$

Then for any scalar c*,*

$$T(c\vec{v}) = 5c\vec{v} = c5\vec{v} = cT(\vec{v})$$

and

$$T(\vec{u} + \vec{v}) = 5(\vec{u} + \vec{v}) = 5\vec{u} + 5\vec{v} = 5T(\vec{u}) + 5T(\vec{v}).$$

Therefore T *is a linear transformation.*

Example 4. *Let* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *be given by*

$$F(x_1, x_2) = (\sin(x_1), e^{x_2}).$$

On the one hand,

$$\begin{aligned} F(0,0) + F(\pi/2, 0) &= (0,1) + (1,1) \\ &= (0,2). \end{aligned}$$

On the other hand

$$\begin{aligned} F((0,0) + (\pi/2, 0)) &= F(\pi/2, 0) \\ &= (0,1). \end{aligned}$$

Since $(0,1) \neq (0,2)$*,* F *is not linear.*

Definition 5. *Let* A *be an* $m \times n$ *matrix. The* matrix transformation T_A *of* A *is the map* $T_A : \mathbb{R}^n \to \mathbb{R}^m$ *defined by*

$$T_A(\vec{X}) = A\vec{x},$$

for any $\vec{x} \in \mathbb{R}^n$*.*

Proposition 6. *For any matrix* A*,* T_A *is a linear transformation.*

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These are lecture notes for HMC Math 40: Introduction to Linear Algebra and roughly follow our course text *Linear Algebra* by David Poole.

PROOF. Let c be a scalar, and $\vec{u}, \vec{v} \in \mathbb{R}^n$. Then

$$T_A(c\vec{v}) = A(c\vec{v}) = cA\vec{v} = cT_A(\vec{v}),$$

and

$$T_A(\vec{u} + \vec{v}) = A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = T(\vec{u}) + T(\vec{v}).$$

□

Theorem 7. *Let* $T : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear transformation. Then* T *is the matrix transformation of the* $m \times n$ *matrix* A *given by*

$$A = \begin{pmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{pmatrix}.$$

PROOF. Let $\vec{x} \in \mathbb{R}^n$. Then

$$\begin{aligned} \vec{x} &= (x_1, \dots, x_n) \\ &= x_1\vec{e}_1 + \cdots + x_n\vec{e}_n. \end{aligned}$$

Therefore

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + \cdots + x_n\vec{e}_n) \\ &= T(x_1\vec{e}_1) + \cdots + T(x_n\vec{e}_n) \\ &= x_1T(\vec{e}_1) + \cdots x_nT(\vec{e}_n) \\ &= \begin{pmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= A\vec{x}. \end{aligned}$$

Definition 8. T_A *is called the standard matrix of* T *and is denoted* $[T]$.

Example 9. *If* $T(\vec{v}) = 5\vec{v}$*, then* $[T] = 5I_n$*, since* $T(\vec{e}_i) = 5\vec{e}_i$.

Theorem 10. *If* S *is a linear transformation from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$ *and* T *is a linear transformation from* $\mathbb{R}^m$ *to* $\mathbb{R}^k$*, then* $S \circ T$ *is a linear transformation from* $\mathbb{R}^n$ *to* $\mathbb{R}^k$ *and*

$$[S \circ T] = [S][T].$$

Definition 11. *Let* $S, T : \mathbb{R}^n \to \mathbb{R}^n$ *be linear transformations. Then* S *and* T *are* inverse transformations *if*

$$S \circ T = I_n = T \circ S.$$

We say that S *and* T *are* invertible transformations.

Theorem 12. *Let* T *be an invertible linear transformation. Then* $[T]$ *is an invertible matrix, and*

$$[T^{-1}] = [T]^{-1}.$$

Definition 13. *An invertible linear transformation is called an* isomorphism.

Theorem 14. *For any linear transformation,*

$$T(\vec{0}) = \vec{0}.$$

PROOF.

$$T(\vec{0}) = T(\vec{0} + \vec{0}) = T(\vec{0}) + T(\vec{0}).$$

□

Remark 15. Why are linear transformations called linear transformations? Perhaps this is easiest to see in $\mathbb{R}^1$. Let $T : \mathbb{R} \to \mathbb{R}$ be a linear transformation. The it is of the form $T(x) = ax$ for some scalar a. This is automatic from the theorem above, since any 1×1 matrix is simply a scalar.

But we can also derive this result from first principles. Let

$$T(1) = a.$$

Then

$$a = T(1) = T\left(\left(\frac{1}{x}\right)x\right) = \frac{1}{x}T(x).$$

Thus $T(x) = a(x)$ for all $x \neq 0$. But $T(0) = 0$ by the above, and thus $T(0) = 0 = 5 \cdot 0$. Thus $T(x) = ax$ for all $x \in \mathbb{R}$. Thus the graph of T is a line through the origin of slope a.