# MATH 40 LECTURE 12: DETERMINANTS

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We've seen that the determinant of the $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

And what's more, we've seen that $A$ is invertible if and only if $\det(A) \neq 0$.

How does this work in higher dimensions? Basically, the compute the determinant of a big matrix by reducing back down to the $2 \times 2$ case. We need some terminology here.

**Definition 1.** *Let $A$ be an $n \times n$ matrix. The $(i, j)$-*minor *of $A$ is the $(n-1) \times (n-1)$ matrix obtained by deleting the $i^{th}$ row and $j^{th}$ column of $A$. It is denoted $A_{ij}$*

**Example 2.** *If $A = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 0 & 10 \\ 0 & 1 & 6 \end{pmatrix}$, then*

$$A_{21} = \begin{pmatrix} 2 & 3 \\ 1 & 6 \end{pmatrix}.$$

**Definition 3.** *Let $A = (a_{ij})$ be an $n \times n$ matrix. The* determinant *of $A$ is*

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}).$$

**Example 4.**

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 \\ 7 & 0 & 10 \\ 0 & 1 & 6 \end{vmatrix} &= (-1)^{(1+1)}(1) \begin{vmatrix} 0 & 10 \\ 1 & 6 \end{vmatrix} + (-1)^{(1+2)}(2) \begin{vmatrix} 7 & 10 \\ 0 & 6 \end{vmatrix} + (-1)^{(1+3)}(3) \begin{vmatrix} 7 & 0 \\ 0 & 1 \end{vmatrix} \\ &= (1)(1)(0-10) + (-1)(2)(42-0) + (1)(3)(7-0) \\ &= -10 - 84 + 21 \\ &= -73. \end{aligned}$$

It turns out that there is nothing special about the minors coming from the first row. We can compute the determinant by expanding in terms of minors of any row or any column.

**Definition 5.** *The $(i, j)$-*cofactor *$C_{ij}$ of $A$ is*

$$C_{ij} = (-1)^{i+j} \det(A_{ij}).$$

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These are lecture notes for HMC Math 40: Introduction to Linear Algebra and roughly follow our course text *Linear Algebra* by David Poole.

**Remark 6.**

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j}.$$

**Theorem 7** (Laplace Cofactor Expansion Theorem). *Let $A$ be an $n \times n$ matrix with $n \geqslant 2$. Then*

$$\begin{aligned}\det(A) &= \sum_{j=1}^{n} a_{ij} C_{ij} \text{ (cofactor expansion along } i^{th} \text{ row)} \\ &= \sum_{i=1}^{n} a_{ij} C_{ij} \text{ (cofactor expansion along } j^{th} \text{ column).}\end{aligned}$$

**Example 8.**

$$\begin{aligned}\det(A) &= (-1)^{(1+2)}(2) \begin{vmatrix} 7 & 10 \\ 0 & 6 \end{vmatrix} + (-1)^{(2+2)}(0) \begin{vmatrix} 1 & 3 \\ 0 & 6 \end{vmatrix} + (-1)^{(3+2)}(1) \begin{vmatrix} 1 & 3 \\ 7 & 10 \end{vmatrix} \\ &= (-2)(42) + 0 + (-1)(10 - 21) \\ &= -84 + 11 \\ &= -73.\end{aligned}$$

**Theorem 9.** *Let $A$ be a square matrix.*

(1) *If $A$ has a zero row or a zero column, then* $\det(A) = 0$.
(2) *If $B$ is obtained from $A$ by interchanging two rows (or two columns), then* $\det(B) = -\det(A)$.
(3) *If $A$ has two identical rows (or columns), then* $\det(A) = 0$.
(4) *If $B$ is obtained by multiplying a single row (or column) of $A$ by the scalar $k$, then* $\det(B) = k\det(A)$.
(5) *If $B$ is obtained by adding a multiple of one row (or column) of $A$ to another, then* $\det(B) = \det(A)$.

**Example 10.**

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

**Corollary 11.** *If $A$ is an $n \times n$ matrix and $k$ is a scalar, then* $\det(kA) = k^n \det A$.

**Corollary 12.** *Let $A$ be an $n \times n$ matrix and let $E$ be an elementary matrix. Then*

$$\det(EA) = \det(E)\det(A).$$

**Theorem 13.** *The square matrix $A$ is invertible if and only if* $\det(A) \neq 0$.

PROOF. Let $R$ be the reduced row echelon form of $A$ and let $E_1, \ldots, E_k$ be the elementary matrices such that

$$E_k E_{k-1} \cdots E_1 A = R.$$

Then

$$\begin{aligned}\det(R) &= \det(E_k E_{k-1} \cdots E_1 A) \\ &= (\det(E_k))(\det(E_{k-1})) \cdots (\det(E_1))(\det(A)).\end{aligned}$$

But the determinant of any elementary matrix is nonzero. Thus

$$\det(A) = 0 \Leftrightarrow \det(R) = 0.$$

But $A$ is invertible if and only if $R = I_n$ by the FTIM. Since $\det(I_n) = 1 \neq 0$, the theorem holds. □

**Theorem 14.** *Let $A$ and $B$ be square matrices of the same size.*

(a) $\det(AB) = \det(A)\det(B)$
(b) $\det A^T = \det A$.
(c) *If $A$ is invertible, then* $\det(A^{-1}) = \dfrac{1}{\det A}$

**Definition 15.** *The* adjoint *of $A$ is the transpose of the matrix of cofactors of $A$.*

$$\textit{adjoint}(A) = (C_{ij})^T.$$

**Theorem 16.** *Let $A$ be an invertible matrix. Then*

$$A^{-1} = \frac{1}{\det A}\textit{adjoint}(A).$$