MATH 40 LECTURE 13: EIGENVALUES AND EIGENVECTORS

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Definition 1. *Let A be an $n \times n$ matrix. If*

$$A\vec{x} = \lambda\vec{x}$$

for a scalar λ and nonzero vector $\vec{x}$ in $\mathbb{R}^n$, then λ is called an eigenvalue *of A and $\vec{x}$ is called an* eigenvector *of A corresponding to λ.*

Example 2. *Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$, and consider $\vec{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$. Note that*

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}.$$

Therefore

$$A\vec{x} = 2\vec{x}.$$

Thus $\vec{x}$ is an eigenvector of A with eigenvalue 2.

Remark 3. Note that $A\vec{x} = \lambda\vec{x}$ if and only if $(A - \lambda I)\vec{x} = \vec{0}$. Thus $\vec{x}$ is an eigenvector of A with eigenvalue λ if and only if $\vec{x}$ is in the null space of $A - \lambda I$. But the null space of $A - \lambda I$ is nontrivial if and only if its determinant is nonzero! Thus, the process of finding eigenvalues and eigenvectors boils down to finding scalars λ for which $\det(A - \lambda I)$ **is** equal to zero!

Definition 4. *The* characteristic polynomial *of the square matrix A is $\det(A - \lambda I)$, and the* characteristic equation *of A is $\det(A - \lambda I) = 0$.*

Example 5. *Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$. The characteristic equation of A is $\det(A - \lambda I) = 0$. We solve for λ.*

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 1-\lambda & 2 \\ 1 & 0-\lambda \end{pmatrix} \\ &= (1-\lambda)(0-\lambda) - 2 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1). \end{aligned}$$

Therefore $\det(A - \lambda I) = 0$ if and only if $\lambda = 2$ or $\lambda = -1$. Thus, the eigenvalues of A are 2 and -1.

Definition 6. *Let λ be an eigenvalue of A. The* eigenspace E_λ *of λ is the set of all eigenvectors of A with eigenvalue λ, together with the zero vector, ie*

$$E_\lambda = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \lambda\vec{x}\}.$$

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These are lecture notes for HMC Math 40: Introduction to Linear Algebra and roughly follow our course text *Linear Algebra* by David Poole.

Example 7. *Again consider* $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$. *To find* E_2*, we must find vectors the null space of* $A - 2I$. *We form the augmented matrix and compute*

$$\left(\begin{array}{cc|c} 1-2 & 2 & 0 \\ 1 & 0-2 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 - R_1} \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Thus, all solutions of this matrix are of the form $-x + 2y = 0$. *Therefore,*

$$\begin{aligned} E_2 &= \text{null}(A - 2I) \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x = 2y \right\} \\ &= \left\{ \begin{pmatrix} 2y \\ y \end{pmatrix} : y \in \mathbb{R} \right\} \\ &= \left\{ y \begin{pmatrix} 2 \\ 1 \end{pmatrix} : y \in \mathbb{R} \right\} \\ &= \text{span} \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

Theorem 8. *The square matrix* A *is invertible if and only if* 0 *is not an eigenvalue of* A.

PROOF. By the FTIM, A is invertible if and only if $A\vec{x} = \vec{0}$ has only the trivial solution. But $A\vec{x} = \vec{0}$ has only the trivial solution if and only if 0 is not an eigenvalue of A. □