MATH 40 LECTURE 14: MORE EIGENVALUES AND EIGENVECTORS

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Theorem 1. *Let A be a square matrix, let $\vec{x}$ be an eigenvector of A with eigenvalue λ. For any $n \in \mathbb{N}$, $\vec{x}$ is an eigenvalue of A^n with eigenvalue λ^n. If A is invertible, this holds for all $n \in \mathbb{Z}$.*

Remark 2. Let A be as above. If A is invertible, the $1/\lambda$ is an eigenvalue of A^{-1}.

PROOF. First, consider the case $n \in \mathbb{N}$. We prove by induction. The base case $A\vec{x} = \lambda\vec{x}$ holds by assumption. Now assume $A^k\vec{x} = \lambda^k\vec{x}$. Then we compute

$$A^{k+1}\vec{x} = A(A^k\vec{x}) = A(\lambda^k\vec{x}) = \lambda^k(A\vec{x}) = \lambda^k(\lambda\vec{x}) = \lambda^{k+1}\vec{x}.$$

Therefore the result holds for $n > 0$. The case $n = 0$ is trivial.

Now consider $n = -1$. We compute

$$\vec{x} = I\vec{x} = (A^{-1}A)\vec{x} = A^{-1}(A\vec{x}) = A^{-1}(\lambda\vec{x}) = \lambda(A^{-1}\vec{x}).$$

Since the $n = -1$ case assumes A^{-1} exists, A is invertible therefore 0 is not an eigenvalue of A. Therefore $\lambda \neq 0$. Therefore we can solve the above equation, yielding

$$A^{-1}(\vec{x}) = \frac{1}{\lambda}\vec{x}.$$

Thus the result holds if $n = -1$.

It remains to establish the result in case $n < 0$. But this follows from an induction argument similar to the one above. □

Example 3. *Again, let $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$. Let $\vec{x} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$. Compute $A^4\vec{x}$.*

We use eigenvectors and eigenvalues. As above, the eigenvalues are 2 and -1. These have eigenvectors $\vec{v}_1 = (2, 1)$ and $\vec{v}_2(-1, 1)$ respectively. Note that these eigenvectors are linearly independent, and thus span $\mathbb{R}^n$. In particular, we see

$$\begin{pmatrix} 3 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

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These are lecture notes for HMC Math 40: Introduction to Linear Algebra and roughly follow our course text *Linear Algebra* by David Poole.

Therefore

$$\begin{aligned} A^4\vec{x} &= A^4(2\vec{v}_1 + \vec{v}_2) \\ &= 2A^4\vec{v}_1 + A^4\vec{v}_2 \\ &= 2(2^4)\vec{v}_1 + (-1)^4\vec{v}_2 \\ &= 32\vec{v}_1 + \vec{v}_2 \\ &= 32\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 63 \\ 33 \end{pmatrix}. \end{aligned}$$

Theorem 4. *Eigenvectors corresponding to distinct eigenvalues are linearly independent.*

Remark 5. The above example shows that it is often very convenient to write a vector as a linear product of the eigenvectors of A. By the above theorem, if an $n \times n$ matrix has n distinct eigenvalues, then they must form a **basis** of $\mathbb{R}^n$. Therefore any vector can be written as a linear combination of the eigenvectors. The general idea here is that of *coordinates*.

Definition 6. *Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis of the subspace $S \subset \mathbb{R}^n$. Then any vector $\vec{x} \in S$ can be written as a unique linear combination*

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k.$$

The vector

$$[\vec{v}]_{\mathcal{B}} = (c_1, \dots, c_k)$$

is called the coordinate vector *of $\vec{v}$ in basis $\mathcal{B}$.*

Example 7. *Let $\mathcal{B}_1 = \{\vec{e}_1, \vec{2}\}$ be the standard basis of $\mathbb{R}^2$, and let*

$$[\vec{v}]_{\mathcal{B}_1} = (3, 3).$$

Now consider the basis $\mathcal{B}_2 = \{\vec{v}_1 = (2, 1), \vec{v}_2 = (-1, 1)\}$. In the above example, we have

$$[\vec{v}]_{\mathcal{B}_2} = (2, 1).$$

Definition 8. *Let λ be an eigenvalue of A. The* algebraic multiplicity *of λ is the degree of λ as a root of the characteristic polynomial. The* geometric multiplicity *of λ is the dimension of the eigenspace E_λ.*

Example 9. *Consider the identity matrix I_2. The characteristic equation of I_2 is*

$$\det(I - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2.$$

Therefore $\lambda = 1$ is the only eigenvalue of I, and it is an eigenvalue of algebraic multiplicity 2.

On the other hand, $I\vec{x} = 1\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$, and therefore $E_\lambda = \mathbb{R}^2$. Thus $\lambda = 1$ has geometric multiplicity 2.