MATH 40 LECTURE 15: DIAGONALIZATION

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In this lecture, we introduce the notion of *diagonalizability* of a square matrix, and relate this new notion to that of eigenvectors and eigenvalues.

Definition 1. *Let λ be an eigenvalue of A. The* algebraic multiplicity *of λ is the degree of λ as a root of the characteristic polynomial. The* geometric multiplicity *of λ is the dimension of the eigenspace E_λ.*

Example 2. *Consider the identity matrix I_2. The characteristic equation of I_2 is*

$$\det(I - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2.$$

Therefore $\lambda = 1$ is the only eigenvalue of I, and it is an eigenvalue of algebraic multiplicity 2.

On the other hand, $I\vec{x} = 1\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$, and therefore $E_\lambda = \mathbb{R}^2$. Thus $\lambda = 1$ has geometric multiplicity 2.

Definition 3. *Let A and B be $n \times n$ matrices. We say that A and B are* similar *if there is an invertible matrix P such that*

$$A = PBP^{-1}.$$

We write $A \sim B$.

Example 4. *Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$. Then A is similar to $B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$. Indeed,*

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}.$$

Theorem 5. *Similarity of matrices is an equivalence relation. In more detail, let A, B and C be $n \times n$ matrices. Then*

(a) $A \sim A$ *(reflexive)*
(b) *If $A \sim B$ then $B \sim A$. (symmetric)*
(c) *If $A \sim B$ and $B \sim C$, then $A \sim C$. (transitive)*

PROOF.

(a) $A = IAI^{-1}$.
(b) Suppose $A \sim B$. Then there exists an invertible P such that $A = PBP^{-1}$. Let $Q = P^{-1}$. Then $B = P^{-1}AP$, so $B = QAQ^{-1}$. Therefore $B \sim A$.
(c) Suppose $A = PBP^{-1}$ and $B = QCQ^{-1}$. Then

$$A = P(QCQ^{-1})P^{-1} = (PQ)C(PQ)^{-1}.$$

Thus $A \sim C$. □

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These are lecture notes for HMC Math 40: Introduction to Linear Algebra and roughly follow our course text *Linear Algebra* by David Poole.

Theorem 6. *If A and B are similar, then A and B have the same determinant, rank and characteristic polynomial.*

Corollary 7. *If A and B are similar, then they have the same eigenvalues, and A is invertible if and only if B is invertible.*

PROOF (of Theorem 6). Since $A \sim B$, there is an invertible P such that $A = PBP^{-1}$. Then

$$\begin{aligned}
\det(A) &= \det(PBP^{-1}) \\
&= \det(P)\det(B)\det(P^{-1}) \\
&= \det(P)\det(B)\frac{1}{\det(P)} \\
&= \frac{\det(P)}{\det(P)}\det(B) \\
&= \det(B).
\end{aligned}$$

Therefore $\det(A) = \det(B)$.

Similarly, (pun intended), we compute

$$\begin{aligned}
\det(A - \lambda I) &= \det(PBP^{-1} - \lambda I) \\
&= \det(PBP^{-1} - \lambda(PIP^{-1})) \\
&= \det[P(B - \lambda I)P^{-1}] \\
&= \det(P)\det(B - \lambda I)\det(P^{-1}) \\
&= \det(P)\det(B - \lambda I)\frac{1}{\det(P)} \\
&= \det(B - \lambda I).
\end{aligned}$$

Therefore A and B have the same characteristic polynomial.

Finally, note that the $\text{nullity}(P) = 0$, since the null space of any invertible matrix is trivial, by the Fundamental Theorem of Invertible Matrices. In other words $PBP^{-1}\vec{x} = \vec{0}$ if and only if $PB\vec{x} = 0$, and $PB\mathbf{x} = 0$ if and only if $B\vec{x} = 0$. Therefore

$$\text{nullity}(A) = \text{nullity}(PBP^{-1}).$$

Therefore, by the Rank-Nullity Theorem, $\text{rank}(A) = \text{rank}(B)$. □

Definition 8. *The matrix A is* diagonalizable *if A is similar to a diagonal matrix.*

Example 9. *The matrix $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ is diagonalizable since*

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}.$$

Theorem 10. *Let A be an $n \times n$ matrix. A is diagonalizable if and only if A has n linearly independent eigenvectors.*

In fact, there is an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$ if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are the corresponding eigenvalues (in the same order).

PROOF. Suppose $A = PDP^{-1}$. Let the columns of P be $\vec{p}_1, \ldots, \vec{p}_n$, and let the diagonal entries of D be $\lambda_1, \ldots, \lambda_n$. Then $AP = PD$, ie

$$A \begin{pmatrix} \vec{p}_1 & \vec{p}_2 & \cdots & \vec{p}_n \end{pmatrix} = \begin{pmatrix} \vec{p}_1 & \vec{p}_2 & \cdots & \vec{p}_n \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

in other words

$$\begin{pmatrix} A\vec{p}_1 & A\vec{p}_2 & \cdots & A\vec{p}_n \end{pmatrix} = \begin{pmatrix} \lambda_1\vec{p}_1 & \lambda_2\vec{p}_2 & \cdots & \lambda_n\vec{p}_n \end{pmatrix}.$$

Therefore

$$A\vec{p}_i = \lambda_i\vec{p}_i$$

for all $1 \leqslant i \leqslant n$. Therefore the columns of P are eigenvectors of A with eigenvalues corresponding to the diagonal entries of D. Moreover, since P is invertible, by FTIM it has linearly independent columns.

Now, assume A has n linearly indepdenent eigenvectors $\vec{p}_1, \ldots, \vec{p}_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively. We prove A is diagonalizable. Indeed, we have

$$\begin{aligned} A\vec{p}_1 &= \lambda_1\vec{p}_1 \\ A\vec{p}_2 &= \lambda_1\vec{p}_2 \\ &\vdots \\ A\vec{p}_n &= \lambda_1\vec{p}_n. \end{aligned}$$

Now, let P be the matrix with columns $\vec{p_1}, \ldots, \vec{p}_n$, and let $D = I(\lambda_1, \ldots, \lambda_n)^T$. Then $AP = PD$ as above. Therefore A is similar to the diagonal matrix D. $\square$