MATH 40 LECTURE 2: PROJECTIONS AND PLANES

DAGAN KARP

In our last lecture we explored the dot product. We begin with a continuation of this exploration.

1. Dot Product andProjections

Theorem 1. *For any vectors $\vec{u}$ and $\vec{v}$, we have*

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \, ||\vec{v}|| \cos\theta,$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.

Proof. Recall the Law of Cosines states that, for the triangle a, b, c,

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

Applying this to the triangle $\vec{u}$, $\vec{v}$, $\vec{u} - \vec{v}$, we have

$$||\vec{u} - \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 - 2||\vec{u}|| \, ||\vec{v}||.$$

Expanding $||\vec{u} - \vec{v}||^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})$ and simplifying gives the result. □

Remark 2. This theorem allows us to compute the angle between any two vectors.

Definition 3. *The vectors $\vec{u}$ and $\vec{v}$ are* orthogonal *if and only if $\vec{u} \cdot \vec{v} = 0$.*

Remark 4. Now that we have control over the angle between two vectors, we can determine when two vectors are parallel. Thus, it makes sense to speak of "the direction" of a vector $\vec{v}$. Any scalar multiple of $\vec{v}$ is parallel to $\vec{v}$. But "the direction" is unique. So, we would like to single out a special vector that is parallel to $\vec{v}$ and captures the information of "the direction" of $\vec{v}$. This is accomplished by the *unit vector* in the direction of $\vec{v}$.

Definition 5. *For any nonzero vector $\vec{u}$, the* unit vector in the direction of $\vec{u}$ *is*

$$\frac{\vec{u}}{||\vec{u}||}.$$

Remark 6. Note that the denominator is nonzero, and this vector is of length one, and is a scalar multiple of $\vec{u}$.

Now that we understand the angle between two vectors, it brings up a natural geometric question. Suppose $\vec{u}$ and $\vec{v}$ are not necessarily pointed in the same direction. How much of $\vec{v}$ is pointed in the direction of $\vec{u}$? In other words, if we orthogonally project $\vec{v}$ in the direction of $\vec{u}$, what do we get?

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These are lecture notes for HMC Math 40: Introduction to Linear Algebra and roughly follow our course text *Linear Algebra* by David Poole.

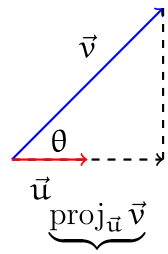

FIGURE 1. The projection of $\vec{v}$ in the direction of $\vec{u}$

Theorem 7. *For a nonzero vector $\vec{u}$, the projection of $\vec{v}$ in the direction of $\vec{u}$ is given by*

$$\text{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u}.$$

Remark 8. Note that the projection of $\vec{v}$ in the direction of $\vec{u}$ is indeed a scalar multiple of $\vec{u}$.

PROOF. As illustrated in Figure 1, let θ be the angle between $\vec{u}$ and $\vec{v}$. Then

$$\cos\theta = \frac{||\text{proj}_{\vec{u}} \vec{v}||}{||\vec{v}||}.$$

Thus

$$\begin{aligned} ||\text{proj}_{\vec{u}} \vec{v}|| &= ||\vec{v}||(\cos\theta) \\ &= ||\vec{v}|| \left(\frac{\vec{u} \cdot \vec{v}}{||\vec{u}||||\vec{v}||} \right) \\ &= \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||}. \end{aligned}$$

Since $\text{proj}_{\vec{u}} \vec{v}$ is in the direction of $\vec{u}$, we simply multiply its length by the unit vector in the direction of $\vec{u}$ to determine $\text{proj}_{\vec{u}} \vec{v}$ completely. Thus we have

$$\begin{aligned} \text{proj}_{\vec{u}} \vec{v} &= ||\text{proj}_{\vec{u}} \vec{v}|| \left(\frac{\vec{u}}{||\vec{u}||} \right) \\ &= \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||} \left(\frac{\vec{u}}{||\vec{u}||} \right) \\ &= \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \, ||\vec{u}||} \, \vec{u} \\ &= \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \, \vec{u}. \quad \square \end{aligned}$$

Example 9. *Compute the projection of* $(1, 2, 3)$ *in the directions* $(0, 1, 0)$ *and* $(1, 1, 0)$.

2. PLANES IN SPACE

Recall that a plane is determined by one point and one vector in $\mathbb{R}^3$. How can we think about this? Well, $\mathbb{R}^3$ seems to have only 3 dimensions. Any plane has exactly two dimensions. So, there is one left over. This direction is perpendicular (orthogonal) to the directions in the plane. See Figure 2.

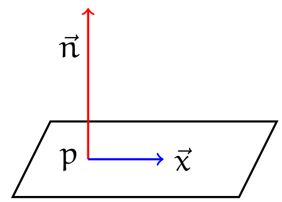

FIGURE 2. Derivation of the normal equation of a plane in $\mathbb{R}^3$

How can we use this to derive an equation for a plane? The equation of a plane completely describes which points $\vec{x} \in \mathbb{R}^3$ are contained in the plane. So, suppose we are given a vector $\vec{n}$ with is orthogonal to every vector in the plane, and a point p on the plane. The vector $\vec{n}$ is called the *normal vector* of the plane.

Let $x \in \mathbb{R}^3$. How can we tell if x is on our plane? Note that the point $\vec{x}$ is in the plane if and only if the vector $x - p$ from p to x is in the plane. Thus, $\vec{x}$ is in the plane if and only if

$$\vec{n} \cdot (x - p) = 0.$$

Let's name the components of our vectors. Let $\vec{n} = (a, b, c)$, let our fixed point p have coordinates $p = (x_0, y_0, z_0)$ and let our general point $\vec{x}$ have coordinates $\vec{x} = (x, y, z)$.

Thus, we have $\vec{x}$ is on our plane if and only if

$$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

Expanding, we have

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

Note that the right hand side $ax_0 + by_0 + cz_0$ is simply a scalar. Call id d. Then we have

$$\boxed{ax + by + cz = d}.$$

This is called the *normal equation* of the plane.

Remark 10. Note that the above equation is *linear*. We see that every plane can be described by a linear equation, and every linear equation describes a plane.

We now see that a plane is determined by a normal vector and a single point on the plane. How can we construct normal vectors? One convenient tool is a construction known as the *cross product*. We will show that given two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$, their cross product $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

Definition 11. *Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{R}^3$. The* cross product *of $\vec{u}$ and $\vec{v}$ is defined by*

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

I find this definition impossible to remember. Instead, there is an easier way to remember it in terms of determinants.

Definition 12. *The* determinant *of the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is*

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The determinant of a 3×3 matrix can be expressed in terms of determinants of 2×2 matrices that appear within the larger matrix.

Definition 13. *The* determinant *of the* 3×3 *matrix* $\begin{pmatrix} a & b & c \\ d & e & f \\ h & i & j \end{pmatrix}$ *is defined by*

$$\begin{vmatrix} a & b & c \\ d & e & f \\ h & i & j \end{vmatrix} = a \begin{vmatrix} e & f \\ i & j \end{vmatrix} - b \begin{vmatrix} d & f \\ h & j \end{vmatrix} + c \begin{vmatrix} d & e \\ h & i \end{vmatrix}.$$

Remark 14. Using the standard notation $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$ and $\vec{k} = (0, 0, 1)$, we see that

$$\begin{aligned} \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} &= \vec{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\ &= (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1). \end{aligned}$$

Proposition 15. $\vec{u} \times \vec{v}$ *is orthogonal to* $\vec{u}$ *and* $\vec{v}$*.*

PROOF. Note that

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\vec{u} \times \vec{v}) = 0. \quad \square$$