MATH 40 LECTURE 3: INTRODUCTION TO SYSTEMS OF LINEAR EQUATIONS

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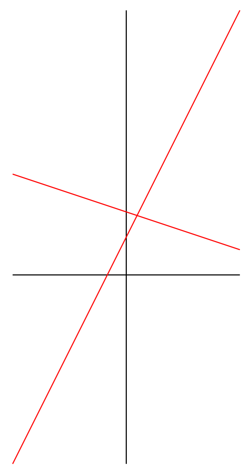
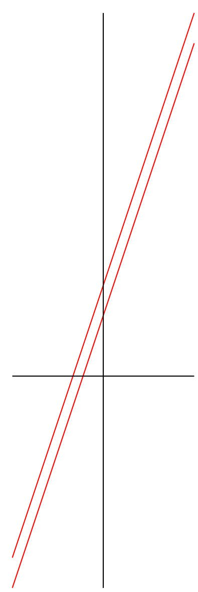
What is the intersection of two lines in the real plane $\mathbb{R}^2$? In other words, how many points are in the intersection of two lines in the plane? The answer, of course, is zero, or one, or infinity!

Example 1. *The lines* $y - 2x = 1$ *and* $3y + x = 5$ *meet at a single point,*



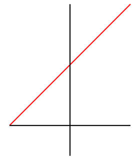
while the lines $y = 3x + 2$ *and* $y = 3x + 3$ *meet at no points,*



Date: January 22, 2012.

These are lecture notes for HMC Math 40: Introduction to Linear Algebra and roughly follow our course text *Linear Algebra* by David Poole.

and the lines $y - x = 1$ and $2y - 2x = 2$ meet at infinitely many points.



Definition 2. *A* linear equation *in two variables x and y is of the form*

$$ax + by = c,$$

where a, b and c are constants.

Definition 3. *A system of linear equations in is* consistent *if it has at least one solution. Otherwise, it is* inconsistent.

Example 4. *The system of equations*

$$\begin{aligned} y - 2x &= 1 \\ 3y + x &= 5 \end{aligned}$$

is consistent.

What is the general situation for a pair of lines in the plane? How can we analyze it? Well the general system of two linear equations in the plane is

$$\begin{aligned} ax + by &= c \\ dx + ey &= f, \end{aligned}$$

where a, b, c, d, e, f are given constants. This system is consistant if it has a solution. Let's try to solve it!

First, if $a = b = 0$, then we only have one equation. So, assume $a \neq 0$. (The proof is the same if we assume $b \neq 0$.) Then

$$x = \frac{1}{a}(c - by).$$

Now that I've solved for x, I'd like to substitute in the second equation.

Again, we have cases. If $d = 0$, then

$$y = f/e \qquad\qquad x = \frac{1}{a}\left(c - \frac{bf}{e}\right) = \frac{1}{ae}(ce - bf).$$

On the other hand, if $d \neq 0$, then

$$\frac{d}{a}(c - by) + ey = f.$$

Thus

$$\frac{cd}{a} + \frac{ae - bd}{a}y = f.$$

Therefore, if $ae - bd \neq 0$, we have

$$y = \frac{af - cd}{ae - bd}.$$

Remark 5. These terms $ae - bd$, $ce - bf$ and $af - cd$ are special. Have you seen them before?

Definition 6. *A* 2×2 real matrix A *is an array of real numbers*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Definition 7. *The* determinant *of the* 2×2 *matrix* A *is*

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Remark 8. We can use matrices to solve systems of linear equations!

Definition 9. *The* augmented matrix *of the system of linear equations*

$$\begin{aligned} ax + by &= c \\ dx + ey &= f, \end{aligned}$$

is given by

$$\left(\begin{array}{cc|c} a & b & c \\ d & e & f \end{array}\right).$$

How do matrices help us solve systems of equations? There are **three allowable moves**.

(1) Interchange two rows.
(2) Multiply one row by a nonzero scalar.
(3) Add a multiple of one row to another row.

Example 10. *Where do the lines* $y - 2x = 1$ *and* $3y + x = 5$ *intersect? We form the augmented matrix*

$$\left(\begin{array}{cc|c} -2 & 1 & 1 \\ 1 & 3 & 5 \end{array}\right)$$

and begin using our allowable moves. First, we multiply the top row by $-1/2$ *to obtain*

$$\left(\begin{array}{cc|c} 1 & -1/2 & -1/2 \\ 1 & 3 & 5 \end{array}\right).$$

Then we add -1 *times Row 1 to Row 2.*

$$\left(\begin{array}{cc|c} 1 & -1/2 & -1/2 \\ 0 & 7/2 & 11/2 \end{array}\right).$$

Continuing,

$$\left(\begin{array}{cc|c} 1 & -1/2 & -1/2 \\ 0 & 1 & 11/7 \end{array}\right).$$

Finally, we have

$$\left(\begin{array}{cc|c} 1 & 0 & 4/14 \\ 0 & 1 & 11/7 \end{array}\right).$$

Therefore, translating this equation back into a linear system, we have

$$\begin{aligned} x + 0 &= 4/14 \\ 0 + y &= 11/7. \end{aligned}$$