# MATH 40 LECTURE 4: GAUSSIAN ELIMINATION

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In our last lecture, we were introduced to the notion that matrices are useful tools for solving linear systems in two variables. In this lecture, we extend this to higher dimensions, point out key ideas and features, and prove a theorem which tells us how to detect infinite solutions to our linear system based on its augmented matrix.

**Definition 1.** *A* linear equation in $\mathbb{R}^n$ *is of the form*

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b,$$

*where* $a_1, \ldots, a_n, b$ *are real numbers.*

**Definition 2.** *An* $m \times n$ real matrix *is an array of real numbers with* $m$ *rows and* $n$ *columns.*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

**Definition 3.** *Let*

$$a_{11} x_1 + a_{21} x_2 + \cdots + a_{1n} x_n = b_1$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = b_m$$

*be a system of* $m$ *linear equations in* $\mathbb{R}^n$. *It's* augmented matrix *is*

$$\left( \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right)$$

**Remark 4.** Note that it is easiest to solve a linear system if its augmented matrix is in a nice and friendly form.

**Definition 5.** *A matrix is in* row echelon form *if*

(1) *any all-zero rows are at the bottom; and*
(2) *in each nonzero row, the first nonzero entry (leading term) is in a column to the left of any leading terms below it.*

**Definition 6.** *A matrix is in* reduced row echelon form *if*

(1) *it is in row echelon form;*

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(2) *the leading entry in each nonzero row is 1; and*

(3) *each column containing a leading 1 has zeros everywhere else.*

**Definition 7.** Gaussian Elimination *is the process of solving a linear system by forming its augmented matrix, reducing to reduced row echelon form, and solving the equation (if the system is consistent).*

**Example 8.** *Let's solve the linear system*

$$x + y - 2z = 0$$
$$2x + 2y - 3z = 1$$
$$3x + 3y + z = 7.$$

*We use Gaussian elimination. First we form the augmented matrix*

$$\left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & 2 & -3 & 1 \\ 3 & 3 & 1 & 7 \end{array} \right).$$

*Now, we reduce to reduced row echelon form using allowable moves (elementary row operations).*

$$\left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & 2 & -3 & 1 \\ 3 & 3 & 1 & 7 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 3 & 3 & 1 & 7 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 7 & 7 \end{array} \right) \rightarrow \left( \begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

*Therefore, we have*

$$x + y - 2z = 0$$
$$z = 1.$$

**Definition 9.** *Variables corresponding to leading terms are called* leading variables. *The other variables are* free variables.

**Example 10.** *In Example 8 above, the leading terms occur in positions $(1,1)$ and $(2,3)$. The corresponding variables are $x$ and $z$. Thus $x$ and $z$ are leading variables in this linear system, and $y$ is a free variable.*

**Remark 11.** Note that Gaussian elimination allows one to solve for leading variables in terms of the free variables. In Example 8, we have

$$x = -y - 2$$
$$z = 1.$$

**Definition 12.** *Two matrices $A$ and $B$ are* row equivalent *if there is a sequence of elementary row operations taking $A$ to $B$.*

**Theorem 13.** *Two matrices $A$ and $B$ are row equivalent if and only if they can be reduced to the same row echelon form.*

**Definition 14.** *The* rank *of a matrix is the number of nonzero rows in its row echelon form.*

**Theorem 15** (Rank Theorem)**.** *Let $A$ be the coefficient matrix of a system of linear equations in $n$ variables. If the system is consistent, then*

$$\text{number of free variables } = n - \text{rank}(A).$$

**Example 16.** *In Example 8, we have*
$$3 - 2 = 1.$$

**Definition 17.** *A system of linear equations is called* homogeneous *if the constant term in each equation is zero.*

**Theorem 18.** *A homogeneous system of* $m$ *linear equations in* $n$ *variables has infinitely many solutions if* $m < n$.

PROOF. The system is consistent since the point $(0,\ldots,0)$ is a solution to any homogeneous equation. The augmented matrix $(A|0)$ has $m$ rows, so the number of nonzero nows is $\leqslant m$. Thus $\text{rank}(A) \leqslant m$. By the Rank Theorem,
$$\text{number of free variables} = n - \text{rank}(A) \geqslant n - m > 0.$$

Therefore this system has at least one free variable, and hence infinitely many solutions.