MATH 40 LECTURE 5: LINEAR INDEPENDENCE AND SPAN

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In this lecture we continue our study of linear systems. In particular, we develop further techniques in our use of matrices to solve linear systems. Along the way, we encounter important notions of *spanning sets*, *linear independence*, and the *transpose of a matrix*.

Recall in our last lecture we encountered the Rank Theorem, which tells us that a linear system in $\mathbb{R}^n$, with coefficient matrix A, has $n - \operatorname{rank}(A)$ free variables, if it is consistent.

Clearly we need tools to determine if a system is consistent or not! Our analysis is going to be built upon the fundamental notion of *linear independence*.

Definition 1. *A* linear combination *of vectors* $\vec{v}_1, \dots, \vec{v}_n$ *is a sum*

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n,$$

where $a_1, a_2, \dots, a_n$ *are constants.*

Example 2.

$$(5,7,6) = 5(1,0,0) + 7(0,1,0) + 6(0,0,1).$$
$$(-2,4) = -2(1,0) + 4(0,1).$$

The following theorem is really just an observation, rephrasing our question from consistency into linear combinations.

Theorem 3. *A linear system with augmented matrix* $(A|\vec{b})$ *is consistent if and only if* $\vec{b}$ *is a linear combination of the columns of* A.

PROOF. A solution of the linear system in $\mathbb{R}^n$ with augmented matrix $(A|\vec{b})$ is a point with coordinates $(x_1, \dots, x_n)$ such that

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}. \quad \square$$

Definition 4. *If* $S = \{\vec{v}_1, \dots, \vec{v}_k\}$ *is a set of vectors in* $\mathbb{R}^n$, *then the* span *of* S *is the set of all linear combinations of the elements of* S,

$$\operatorname{span}(S) = \{a_1\vec{v}_1 + \cdots + a_k\vec{v}_k : a_1, \dots, a_k \in \mathbb{R}\}.$$

We say that S *is a* spanning set *of* $\operatorname{span}(S)$.

Example 5. (1) $\mathbb{R}^2 = \operatorname{span}\{(1,0),(1,1)\}$

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These are lecture notes for HMC Math 40: Introduction to Linear Algebra and roughly follow our course text *Linear Algebra* by David Poole.

(2) $\mathbb{R}^3 = \text{span}\{(1,0,0),(0,1,0),(0,0,1)\}$

Definition 6. *In* $\mathbb{R}^n$, *let*

$$\begin{aligned} e_1 &= (1,0,\dots,0) \\ e_2 &= (0,1,0\cdots,0) \\ &\vdots \\ e_n &= (0,0,\dots,0,1). \end{aligned}$$

Remark 7.

$$\mathbb{R}^n = \text{span}\{e_1,\dots,e_n\}.$$

Definition 8. *A set of vectors* $\{\vec{v}_1,\dots,\vec{v}_k\}$ *in* $\mathbb{R}^n$ *is* linearly independent *if*

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

only if

$$c_1 = c_2 = \cdots = c_k = 0.$$

Otherwise they are linearly dependent.

Example 9. *Are the vectors* $\vec{v}_1 = (1,2)$, $\vec{v}_2 = (-1,3)$, $\vec{v}_3 = (2,7)$ *linearly independent? Let's explore. Suppose*

$$a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = (0,0).$$

This holds if and only if

$$\begin{aligned} a\cdot 1 + b\cdot(-1) + c\cdot 7 = 0 \text{ and} \\ a\cdot 2 + b\cdot 3 + c\cdot 7 = 0. \end{aligned}$$

So we are reduced to solving a linear system! We know how to attack this problem; we form the augmented matrix and use Gaussian elimination.

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 2 & 3 & 7 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 5 & 3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 3/5 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 13/5 & 0 \\ 0 & 1 & 3/5 & 0 \end{array}\right).$$

Therefore the system is consistent, with free variable z, *so there infinitely many solutions. They are of the form*

$$\begin{aligned} a &= -\frac{13}{5}c \\ b &= -\frac{3}{5}c. \end{aligned}$$

Therefore the vectors $\vec{v}_1, \vec{v}_2\vec{v}_3$ *are linearly dependent. For example, if* $c = 5$, *we have*

$$-13\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 3\begin{pmatrix} -1 \\ 3 \end{pmatrix} + 5\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Remark 10. The vectors $\vec{v}_1$, $\vec{v}_2$,...,$\vec{v}_k$ in $\mathbb{R}^n$ are linearly dependent if and only if one of the vectors is a linear combination of the others.

Theorem 11. *Let* $\vec{v}_1,\dots,\vec{v}_k$ *be column vectors in* $\mathbb{R}^n$ *and let* A *be the* $n \times k$ *matrix*

$$A = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{pmatrix}.$$

Then $\vec{v}_1,\dots,\vec{v}_k$ *are linearly dependent if and only if the homogeneous linear system with augmented matrix* $(A|\vec{0})$ *has a nontrivial solution.*

Theorem 12. *Let $\vec{v}_1, \dots, \vec{v}_m$ be row vectors in $\mathbb{R}^n$, and let A be the $m \times n$ matrix*

$$A = \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_m \end{pmatrix}.$$

Then $\vec{v}_1, \dots, \vec{v}_m$ are linearly dependent if and only if $\operatorname{rank}(A) < m$.

Corollary 13. *Any set of m vectors in $\mathbb{R}^n$ is linearly dependent if $m > n$.*

Example 14. *Recall our old friend, the linear system*

$$\begin{aligned} x + y - 2z &= 0 \\ 2x + 2y - 3z &= 1 \\ 3x + 3y + z &= 7. \end{aligned}$$

We know this system is consistent if the columns of its augmented matrix are linearly dependent. This augmented matrix is

$$A = \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & 2 & -3 & 1 \\ 3 & 3 & 1 & 7 \end{array}\right).$$

We also know that these four column vectors are independent if the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -2 & -3 & 1 \\ 0 & 1 & 7 \end{pmatrix}$$

has rank less than 4. But this matrix has at least one zero row. Therefore its rows are not linearly independent. Therefore the columns of A are linearly dependent. Therefore our original linear system is consistent.