

MATH 40 LECTURE 6: LINEAR INDEPENDENCE CONTINUED AND MATRIX OPERATIONS

DAGAN KARP

In this lecture we continue our study of linear independence, and then discuss basic matrix operations.

Recall, we ended last time with the following theorem, which characterizes linear independence in terms of a matrix of column vectors.

Theorem 1. Let $\vec{v}_1, \dots, \vec{v}_k$ be column vectors in \mathbb{R}^n and let A be the $n \times k$ matrix

$$A = (\vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_k).$$

Then $\vec{v}_1, \dots, \vec{v}_k$ are linearly dependent if and only if the homogeneous linear system with augmented matrix $(A|\vec{0})$ has a nontrivial solution.

Theorem 2. Let A be an $m \times n$ matrix, and consider the homogeneous linear system with augmented matrix $(A|\vec{0})$. If $m < n$, then the system has infinitely many solutions. (More columns than rows means infinitely many solutions for a homogeneous system.)

PROOF. Any homogeneous system is consistent, since $(x_1, \dots, x_n) = \vec{0}$ is always a solution. Therefore, by the Rank Theorem,

$$\# \text{ free variables} = n - \text{rank}(A) \geq n - m > 0.$$

Therefore the system is consistent with at least one free variable. Thus the system has infinitely many solutions. \square

Corollary 3. Any set of k vectors in \mathbb{R}^n is linearly dependent if $k > n$.

PROOF. Apply the two previous theorems. We may also characterize linear independence in terms of the associated matrix of rows.

Theorem 4. Let $\vec{v}_1, \dots, \vec{v}_m$ be row vectors in \mathbb{R}^n , and let A be the $m \times n$ matrix

$$A = \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_m \end{pmatrix}.$$

Then $\vec{v}_1, \dots, \vec{v}_m$ are linearly dependent if and only if $\text{rank}(A) < m$.

PROOF. The rank of A is less than m if and only if a sequence of elementary row operations results in a zero row. But a sequence of elementary row operations is merely a linear combination of rows. Therefore the $\text{rank}(A) < m$ if and only if one row is a linear combination of the others, ie, the rows are linearly dependent. \square

Date: January 29, 2012.

These are lecture notes for HMC Math 40: Introduction to Linear Algebra and roughly follow our course text *Linear Algebra* by David Poole.

Example 5. Recall our old friend, the linear system

$$\begin{aligned}x + y - 2z &= 0 \\2x + 2y - 3z &= 1 \\3x + 3y + z &= 7.\end{aligned}$$

We know this system is consistent if the columns of its augmented matrix are linearly dependent. This augmented matrix is

$$A = \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & 2 & -3 & 1 \\ 3 & 3 & 1 & 7 \end{array} \right).$$

We also know that these four column vectors are independent if the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -2 & -3 & 1 \\ 0 & 1 & 7 \end{pmatrix}$$

has rank less than 4. But this matrix has at least one zero row. Therefore its rows are not linearly independent. Therefore the columns of A are linearly dependent. Therefore our original linear system is consistent.

Remark 6. Above we used two closely related matrices. If $\vec{v}_1, \dots, \vec{v}_m$ are vectors in \mathbb{R}^n , then we can form the $m \times n$ matrix of columns or the $n \times m$ matrix of rows. The i^{th} column of one is the i^{th} row of the other, and vice versa. This “flipping” of a matrix is important enough to have a special name.

Definition 7. Let $A = (a_{ij})$ be an $m \times n$ matrix. The transpose of A is the $n \times m$ matrix A^T defined by

$$(A^T)_{ij} = a_{ji}.$$

In other words, the number in row i and column j of A^T is the number found in row j column i of A .

Example 8. We compute.

$$\begin{aligned}\bullet & \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \\ \bullet & (1, 0)^T = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\end{aligned}$$

Definition 9. A matrix is symmetric if $A = A^T$.

Remark 10. If A is symmetric, then A has the same number of rows and columns.

Definition 11. Let $A = (a_{ij})$ be an $m \times n$ matrix.

- (1) A is square if $n = m$.
- (2) A is diagonal if $a_{ij} = 0$ when $i \neq j$.
- (3) A is scalar if $a_{ij} = c$ for some scalar c and all i, j .

Example 12. The identity matrix is the square diagonal matrix with all 1's on the diagonal.

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Definition 13. Let $A = (a_{ij})$ and $B = (b_{ij})$ both be $m \times n$ matrices. The sum of A and B is defined by $A + B = C$, where $C = (c_{ij})$ is defined by

$$c_{ij} = a_{ij} + b_{ij}.$$

Definition 14. Let $A = (a_{ij})$ be an $m \times n$ matrix and let $B = (b_{jk})$ be an $n \times l$ matrix. Then the product $A \times B$ is defined by $A \times B = C$, where $C = (c_{ij})$ is given by

$$c_{ij} = \sum_{j=1}^n a_{ij} b_{jk}.$$