MATH 40 LECTURE 7: INVERTIBLE MATRICES

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In this lecture we define what it means for a matrix to be invertible, discuss first properties and examples of invertible matrices, determine criteria for invertibility, and see a deep connection between the inverse of a matrix and the solution to an associated system of linear equations.

Definition 1. *Let* A *be an* $n \times n$ *matrix. The matrix* B *is the* inverse *of* A *if*

$$AB = BA = I,$$

where $I = I_n$ *is the* $n \times n$ *identity matrix. If such a matrix* B *exists, then* A *is called* invertible.

Example 2. *Let* A *be given by*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore the matrix

$$B = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

is an inverse of A.

Example 3. *Is the matrix*

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

invertible? Lets try. If

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

then

$$\begin{aligned} a + 2c &= 1 \\ b + 2d &= 0 \\ a + 2c &= 0 \\ b + 2d &= 1. \end{aligned}$$

This is impossible (because $0 \neq 1$*). Therefore* A *is not invertible.*

Theorem 4. *The inverse of a matrix is unique.*

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These are lecture notes for HMC Math 40: Introduction to Linear Algebra and roughly follow our course text *Linear Algebra* by David Poole.

PROOF. Let A be an invertible matrix. Suppose B and C are inverses of A, so that

$$AB = BA = I \text{ and}$$
$$AC = CA = I.$$

Then we compute

$$B = BI = B(AC) = (BA)C = IC = C. \quad \square$$

Remark 5. We denote the inverse of A by A^{-1}.

Theorem 6. *If A is an $n \times n$ invertible matrix, then the system of linear equations $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.*

PROOF. Note that

$$A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I\vec{b} = \vec{b}.$$

Therefore $\vec{x} = A^{-1}\vec{b}$ is a solution to the equation $A\vec{x} = \vec{b}$.

Now suppose there is another solution $\vec{y}$, so that $A\vec{y} = \vec{b}$. Then $y = A^{-1}\vec{b} = \vec{x}$. $\square$

Theorem 7. *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then A is invertible if and only if*

$$ad - bc \neq 0.$$

If A is invertible, its inverse is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Definition 8. *Any matrix formed by applying a single elementary row operation to the identity matrix is called an* elementary matrix.

Example 9. *The matrix*

$$\begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}$$

is elementary since it is obtained from I_2 by multiplying the second row by 7.

How can we actually compute the inverse of a given matrix?? One technique is given by the following theorem.

Theorem 10 (Gauss-Jordan)**.** *Let A be a square matrix. If a sequence of elementary row operations reduces A to I, then the same sequence of operations transforms I to A^{-1}.*

Remark 11. Thus, we can augment a given matrix A with the identity matrix I forming $(A|I)$, and if we reduce A to I, then the right hand matrix must be A^{-1}.

Example 12. *Let's compute the inverse of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. We form the augmented matrix and compute*

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right)$$

Therefore,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}.$$