MATH 40 LECTURE 8: THE FUNDAMENTAL THEOREM OF INVERTIBLE MATRICES

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In our last lecture we were introduced to the notion of the inverse of a matrix, we used the Gauss-Jordan method to find the inverse of a matrix, and we saw that any linear system with an invertible matrix of coefficients is consistent with a unique solution.Now, we turn our attention to properties of the inverse, and the Fundamental Theorem of Invertible Matrices.

Theorem 1. *The following hold.*

(a) *If A is invertible, then A^{-1} is invertible, and*
$$(A^{-1})^{-1} = A.$$
(b) *If A is invertible and $0 \neq c \in \mathbb{R}$, then cA is invertible and*
$$(cA)^{-1} = \frac{1}{c}(A^{-1}).$$
(c) *If A and B are both invertible matrices of the same size, then AB is invertible and*
$$(AB)^{-1} = B^{-1}A^{-1}.$$
(d) *For any matrices A and B,*
$$(A+B)^T = A^T + B^T, \text{ and}$$
$$(AB)^T = B^T A^T.$$
(e) *If A is invertible, then A^T is invertible and*
$$(A^T)^{-1} = (A^{-1})^T.$$
(f) *If A is an invertible matrix, then A^n is invertible for all $n \in \mathbb{N}$, and*
$$(A^n)^{-1} = (A^{-1})^n.$$

PROOF.

(a) Note that
$$A(A^{-1}) = (A^{-1})A = I.$$
Thus A^{-1} is invertible, with inverse A.

(b) Note that for any matrices X and Y and scalar c, we have
$$c(XY) = (cX)Y = X(cY),$$
whenever the product exists. Thus, we have
$$cA(A^{-1}/c) = c/c(AA^{-1}) = I = (A^{-1}A)c/c = (A^{-1}/c)(cA).$$

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These are lecture notes for HMC Math 40: Introduction to Linear Algebra and roughly follow our course text *Linear Algebra* by David Poole.

(c) We must find a matrix X such that

$$X(AB) = (AB)X = I.$$

We compute

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

Similarly, $ABB^{-1}A^{-1} = I$. Thus AB is invertible with inverse $B^{-1}A^{-1}$.

(d) Let $A = (a_{ij}$ and $B = (b_{ij})$. Then $A^T = (a_{ji})$ and $B^T = (b_{ji})$. Then

$$(A + B)^T = (a_{ij} + b_{ij})^T = (a_{ji} + b_{ji}) = A^T + B^T.$$

Now, note that

$$\begin{aligned}(AB)^T_{ij} &= (AB)_{ji} \\ &= \text{row}_j(A) \cdot \text{column}_i(B) \\ &= \text{column}_j(A^T) \cdot \text{row}_i(B^T) \\ &= \text{row}_i(B^T) \cdot \text{column}_j(A^T) \\ &= (B^T A^T)_{ij}.\end{aligned}$$

(e) Exercise 3.3.15

(f) Induction. □

Theorem 2 (Fundamental Theorem of Invertible Matrices). *Let A be an $n \times n$ invertible matrix. TFAE (The Following Are Equivalent.)*

(1) *A is invertible.*
(2) *$A\vec{x} = \vec{b}$ has a unique solution for every vector $\vec{b} \in \mathbb{R}^n$.*
(3) *$A\vec{x} = \vec{0}$ has only the trivial solution.*
(4) *The reduced row echelon form of A is I_n.*
(5) *A is a product of elementary matrices.*

PROOF. We will prove $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$. We proved $(1) \Rightarrow (2)$ in Lecture 7. Any homogeneous system has the trivial solution, thus $(2) \Rightarrow (3)$. If $A\vec{x} = \vec{0}$ has only the trivial solution, then the augmented matrix $(A|\vec{0})$ has no free variables. Thus there are no zero rows in the reduced row echelon form of A. Therefore the reduced row echelon form of A has n nonzero rows, each with leading term 1, each to the left of those below. The only such matrix is I_n. Applying an elementary row operation to A corresponds to multiplying A by the appropriate elementary matrix. Since A can be reduced to I by elementary row operations, it follows that there are elementary matrices $E_1, \dots, E_r$ such that

$$E_1 \cdots E_r A = I.$$

The result follows by solving for A. Finally, any product of elementary matrices is invertible, and thus $(5) \Rightarrow (1)$. □

Theorem 3. *Let A be an $n \times n$ matrix. If B is an $n \times n$ matrix such that either $BA = I$ or $AB = I$, then A is invertible and $B = A^{-1}$.*

PROOF. Suppose $BA = I$. We will show A is invertible with inverse B. Consider the homogeneous linear system $A\vec{x} = \vec{0}$. Since $BA = I$, we have

$$B(A\vec{x}) = (BA)\vec{x} = I\vec{x} = \vec{x} = \vec{0}.$$

Thus the equation $A\vec{x} = \vec{0}$ has only the trivial solution. Therefore, by the FTIM, A is invertible. Thus A^{-1} exists and

$$B = BI = B(AA^{-1}) = (BA)A^{-1} = IA^{-1} = A^{-1}. \quad \square$$

PROOF (of Gauss-Jordan.) Let $E_1, \dots, E_r$ be elementary matrices corresponding to the elementary row operations transforming A to I. Then

$$E_1 E_2 \cdots E_r A = I.$$

Therefore $A^{-1} = E_1 \cdots E_r$ since inverses are unique by FTIM. Therefore

$$E_1 \cdots E_r I = A^{-1},$$

and thus our sequence of elementary row operations corresponding to E_1 through E_r transform I into A^{-1}.