MATH 40 LECTURE 9: SUBSPACE

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What does it mean for one vector space to live inside of another vector space? For example, isn't every line a copy of $\mathbb{R}^1$? So, isn't any line in $\mathbb{R}^3$ an example of one vector space living inside of another? And how about a plane in three-space?

The answer isn't quite so simple. We need not only to preserve the *set theoretic* properties of the line (all of the points on the line), or the *geometric* properties of the line (straight, not curved, infinite), but we also need to preserve the *algebraic properties* of $\mathbb{R}^1$ (its addition and multiplication). There are two common sense requirements fulfilling this need: every vector space needs to contain the zero vector, and it needs to be closed under linear combinations.

Definition 1. *A collection S of vectors in $\mathbb{R}^n$ is called a* subspace *if*

(1) *the vector $\vec{0}$ is in S; and*
(2) *for any vectors $\vec{u}, \vec{v} \in S$ and scalars $s, t \in \mathbb{R}$, their linear combination is also in S,*

$$s\vec{u} + t\vec{v} \in S.$$

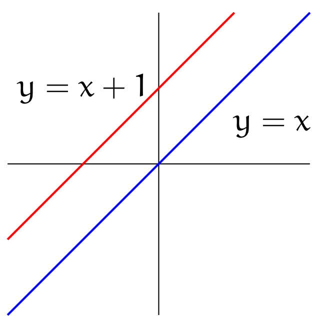

FIGURE 1. A subset of $\mathbb{R}^2$ and a subspace of $\mathbb{R}^2$

Example 2. *Note that $\mathbb{R}^n$ is a subspace of itself. But it is not a* proper *subspace. A subspace S of $\mathbb{R}^n$ is only proper if $S \neq \mathbb{R}^n$. Also, note that $\{\vec{0}\}$ is always a subspace of $\mathbb{R}^n$. It is called the* trivial *subspace.*

Example 3. *Is the line $y = x + 1$ a subspace of $\mathbb{R}^2$? Well, no, it isn't. The set*

$$S = \{(x, y) \in \mathbb{R}^2 : y = x + 1\}$$

satisfies neither of the requirements of a subspace. The point $(0, 0)$ is not in S (it is not on the line). Also, the points $(-1, 0)$ and $(0, 1)$ are both on the line, but the point

$$(-1, 1) = (-1, 0) + (0, 1)$$

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is not on the line.

Remark 4. A line (or plane, or hyperplane) that does not contain the origin is called *affine*.

Remark 5. What is the difference between an affine line and a vector subspace? They look the same, don't they? Well, the have similar geometry but different algebraic structure. Note that an affine line has no origin. So addition doesn't make sense, as we've seen in the example above. This has its own uses. For example, the affine line is often used to model time. I know that the *difference* between 2pm and 3pm (on the same day) is. But what is 2pm *plus* 3pm? The question does not make sense without an origin – a point of reference.

Remark 6. For any collection of vectors $\vec{v}_1, \dots, \vec{v}_k$ in $\mathbb{R}^n$, their span is a vector subspace of $\mathbb{R}^n$. Indeed,

$$\vec{0} = 0\vec{v}_1 + \cdots + 0\vec{v}_k \in \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_k\}$$

and, for any scalars s, t and $\vec{v}_i, \vec{v}_j$,

$$s\vec{v}_i + t\vec{v}_j \in \operatorname{span}\{\vec{v}_1, \dots, \vec{v}_k\},$$

where $1 \leqslant i, j \leqslant k$.

Definition 7. *The* subspace spanned by $\vec{v}_1, \dots, \vec{v}_k$ *is simply the span of* $\{\vec{v}_1, \dots, \vec{v}_k\}$*.*

Definition 8. *Let* A *be an* $m \times n$ *matrix. The* row space *of* A *is the subspace* $\operatorname{row}(A)$ *spanned by the rows of* A*. The* column space *of* A *is the subspace* $\operatorname{col}(A)$ *spanned by the columns of* A*.*

Example 9. *Consider our old friend*

$$A = \begin{pmatrix} 1 & 1 & -2 \\ 2 & 2 & -3 \\ 3 & 3 & 1 \end{pmatrix}.$$

The column space of A *is the set of all* $\vec{b} \in \mathbb{R}^3$ *such that* $\vec{b} \in \operatorname{col}(A)$*. So* $\vec{b} \in \operatorname{col}(A)$ *if and only if* $\vec{b}$ *is a linear combination of the columns of* A*. In other words,* $\vec{b} \in \operatorname{col}(A)$ *if and only if there are scalars* x, y, z *such that*

$$x \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + y \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + z \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}.$$

Therefore

$$\boxed{\vec{b} \in \operatorname{col}(A) \quad \textit{if and only if} \quad A\vec{x} = \vec{b} \textit{ has a solution.}}$$

For example, we previously showed $(0, 1, 7) \in \operatorname{col}(A)$*.*

Remark 10. The rows of A are the columns of A^T. Thus, $\vec{b} \in \operatorname{row}(A)$ if and only if $\vec{b}^T \in \operatorname{col}(A^T)$. Therefore, $\vec{b} \in \operatorname{row}(A)$ if and only if the system

$$A^T\vec{x} = (\vec{b})^T$$

is consistent.

Definition 11. *Let* A *be an* $m \times n$ *matrix. The* null space *of* A*, denoted* $\operatorname{null}(A)$*, is the set of all solutions to the homogeneous system* $A\vec{x} = \vec{0}$*, ie*

$$\operatorname{null}(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}.$$

Example 12. *The null space of*

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

is given by

$$\mathrm{null}(A) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x = 2y \right\}.$$