Big ideas of linear algebra

The basic ingredients are...

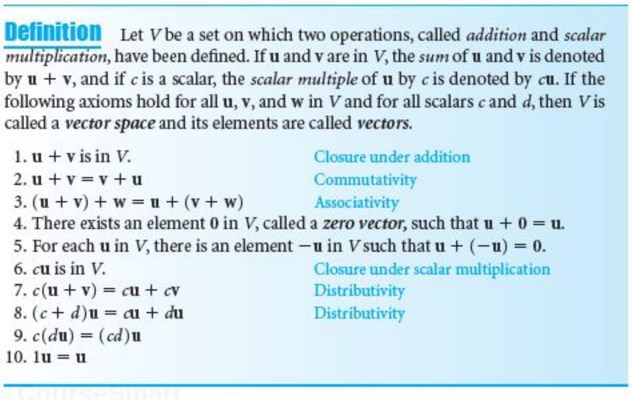
objects
$\Downarrow$
vectors

sets
collections of objects
(w/specific properties)
$\Downarrow$
vector spaces

functions
ways to transform
objects
$\Downarrow$
linear transformations

Heart of linear algebra $\Longrightarrow$ study of linear transformations
and their algebraic properties

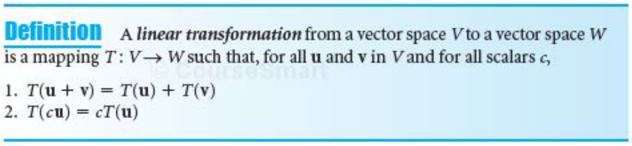
Definition of a vector space (page 447)



Definition Let V be a set on which two operations, called *addition* and *scalar multiplication*, have been defined. If $\mathbf{u}$ and $\mathbf{v}$ are in V, the *sum* of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} + \mathbf{v}$, and if c is a scalar, the *scalar multiple* of $\mathbf{u}$ by c is denoted by $c\mathbf{u}$. If the following axioms hold for all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d, then V is called a ***vector space*** and its elements are called ***vectors***.

1. $\mathbf{u} + \mathbf{v}$ is in V. Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ Commutativity
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ Associativity
4. There exists an element $\mathbf{0}$ in V, called a *zero vector*, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is an element $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V. Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ Distributivity
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ Distributivity
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

Definition of a linear transformation (page 490)



Definition A ***linear transformation*** from a vector space V to a vector space W is a mapping $T: V \rightarrow W$ such that, for all $\mathbf{u}$ and $\mathbf{v}$ in V and for all scalars c,

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$

Remark Note that the def'n of vector space does not say exactly what vectors <u>are</u>, only what vectors <u>do</u>.

$\Longrightarrow$ example of abstraction

exact substance of an object
is not so important, only its
properties and behaviors

powerful technique --- allows us
to simultaneously prove a result
is true in a multitude of contexts

Euclidean vectors – $\mathbb{R}^n$

$\mathbb{R}^n \quad \Longrightarrow \quad$ the set of all ordered n-tuples of real numbers (expressed as column or row vectors)

boldface in textbook

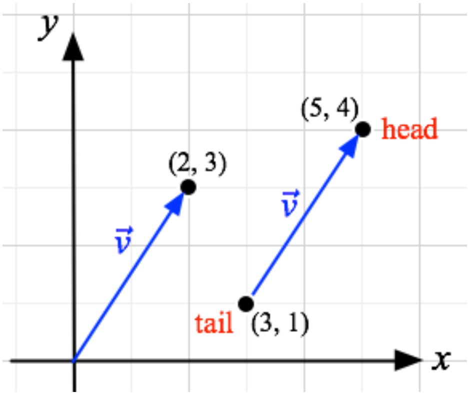
$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

v_i's are components of $\vec{v}$

Euclidean vectors have a length and a direction.

Example

$$\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



vector addition

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

scalar multiplication

$$\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$$

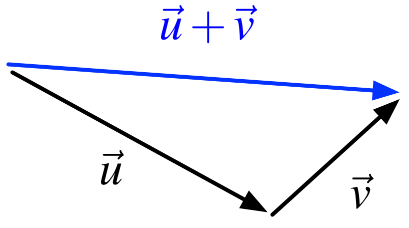
Note: letters at end of alphabet are usually reserved for vectors, and letters at start are usually reserved for scalars.

$\mathbb{R}^n$ with vector addition and scalar multiplication as defined above is a vector space!

Geometric interpretation in $\mathbb{R}^2$

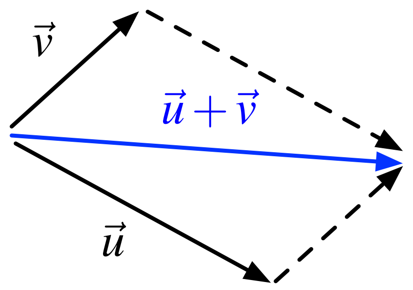
Vector addition:

head-to-tail rule



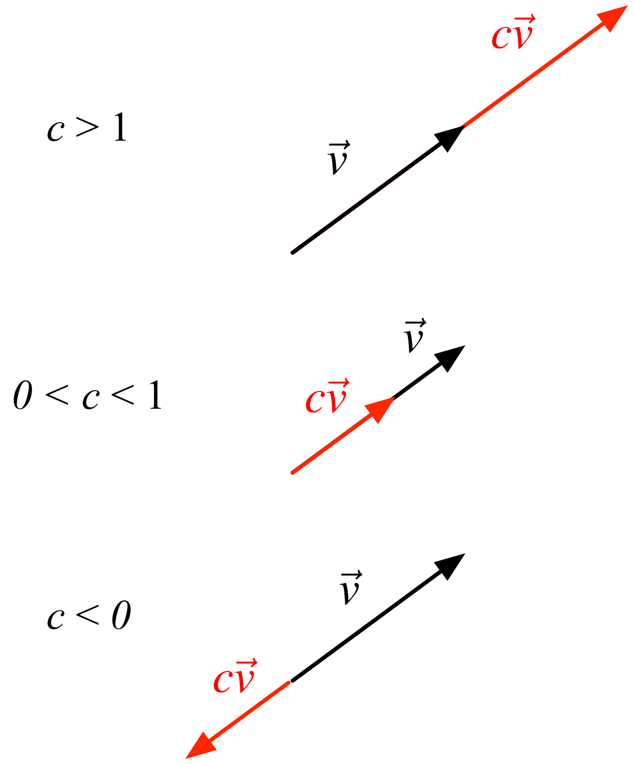
place tail of $\vec{v}$ at head of $\vec{u}$

parallelogram rule



place tails of $\vec{u}$ and $\vec{v}$ at same point

Scalar multiplication:



Food for thought:
What is the other diagonal of the parallelogram in terms of $\vec{u}, \vec{v}$?

Linear combinations

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Revisiting our earlier example, note that

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In this case, we say that $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is a *linear combination* of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Definition A *linear combination* of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is any vector of the form

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_k\vec{v}_k,$$

where $c_1, c_2, \ldots, c_k$ are scalars.

Can we represent $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ by other linear combinations besides the one given above?

$$\text{e.g.,} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 5 \\ 6 \end{bmatrix} + (-1) \begin{bmatrix} 4 \\ 3 \end{bmatrix} + (-4) \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}.$$

Question (to be addressed down the road): Given vector $\vec{v}$ and vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$, is $\vec{v}$ a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$?

Dot product

Geometric concepts of length and orthogonality of vectors in $\mathbb{R}^n$ can be defined algebraically using the dot product.

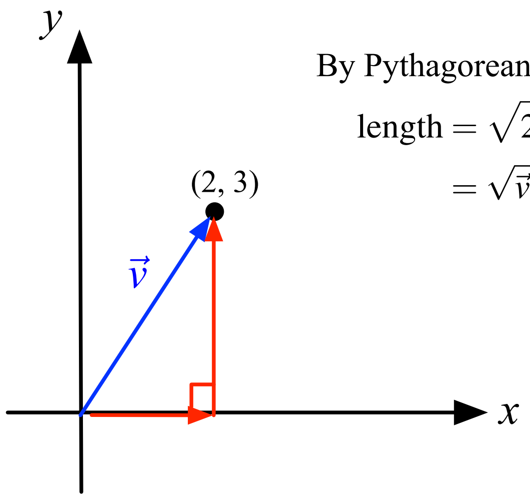
Definition For $\vec{u}, \vec{v} \in \mathbb{R}^n$, the *dot product* of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \cdot \vec{v} = \underbrace{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n}_{}.$$

operation on two vectors that produces a scalar value

To define the length of a vector, think about what it should be for a simple vector in $\mathbb{R}^2$:

$$\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



By Pythagorean Thm,

$$\text{length} = \sqrt{2^2 + 3^2} = \sqrt{\vec{v} \cdot \vec{v}}$$

Definition The *length* of $\vec{v}$ in $\mathbb{R}^n$ is denoted $\|\vec{v}\|$ and is equal to

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \underbrace{\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.}_{\text{always a nonnegative value}}$$

Remarks

- The only vector with length 0 is the zero vector $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

- Length of a scaled vector:

$$\text{for scalar } c,\ \|c\vec{v}\| = \sqrt{(c\vec{v})\cdot(c\vec{v})} = \sqrt{c^2(\vec{v}\cdot\vec{v})} = |c|\,\|\vec{v}\|$$

- Special name for vectors of length 1: *unit vectors*

$$\text{examples: } \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1/2 \\ \sqrt{2}/2 \\ -1/2 \end{bmatrix}$$

- Given any vector in $\mathbb{R}^n$, can we always scale it to get a unit vector in the same direction? (**Yes!** so long as $\vec{v} \neq \vec{0}$)

$$\begin{array}{c}\text{Want to find } c \in \mathbb{R} \text{ such that} \\ c > 0 \text{ and } \|c\,\vec{v}\| = 1\end{array} \quad \Longrightarrow \quad \text{Scale by } c = \frac{1}{\|\vec{v}\|}$$

Example $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ has length $\|\vec{v}\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$, so

$$\frac{1}{\|\vec{v}\|}\vec{v} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

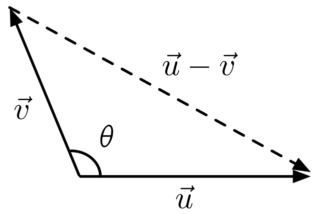
Process above is sometimes referred to as *normalizing* a vector.

Remark Two inequalities regarding lengths of vectors you should see in the text:

- Cauchy-Schwarz Inequality: $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$
- Triangle Inequality: $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

What about the angle between two vectors? (Note: we always assume the angle is between 0 and π.)

In $\mathbb{R}^2$, we can apply Law of Cosines to the triangle



and use fact that $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$ to obtain

$$\underset{\substack{\uparrow\\ \text{algebraic}}}{\vec{u}\cdot\vec{v}} = \underset{\substack{\uparrow\\ \text{geometric}}}{\|\vec{u}\|\|\vec{v}\|\cos\theta}$$

We generalize this to $\mathbb{R}^n$.

For two nonzero vectors $\vec{u},\vec{v}\in\mathbb{R}^n$,

$$\boxed{\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}} \qquad \text{where } \theta \text{ is angle between } \vec{u} \text{ and } \vec{v}.$$

What happens when θ is 90° or $\frac{\pi}{2}$?

$\vec{u}$ and $\vec{v}$ are *orthogonal* $\iff$ angle between $\vec{u}$ and $\vec{v}$ is 90° $\iff$ $\vec{u}\cdot\vec{v}=0$

Notice that

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\boxed{\|\vec{u}\|\|\vec{v}\|}} \longleftarrow \text{always positive}$$

leads to the following:

sign of $\vec{u}\cdot\vec{v}$	angle between $\vec{u}$ and $\vec{v}$
$+$	acute
$-$	obtuse
0	right angle