# Dot product (continued)

For two nonzero vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$\boxed{\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}} \qquad \text{where } \theta \text{ is angle between } \vec{u} \text{ and } \vec{v}.$$

Note that we can use this identity to prove the Cauchy-Schwarz Inequality.

**Theorem** (Cauchy-Schwarz Inequality)**.** *For any vectors* $\vec{u}, \vec{v} \in \mathbb{R}^n$,

$$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\ \|\vec{v}\|.$$

*Proof.* If $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$, then clearly both sides of the inequality are 0 and the result holds.

Assume $\vec{u}, \vec{v} \neq \vec{0}$. By the previous identity,

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\ \|\vec{v}\|\cos\theta \qquad (\star).$$

Thus,

$$\begin{aligned} |\vec{u}\cdot\vec{v}| &= |\ \|\vec{u}\|\ \|\vec{v}\|\cos\theta\ | \quad \text{by } (\star) \text{ above} \\ &= \|\vec{u}\|\ \|\vec{v}\|\ |\cos\theta| \quad \text{since } \|\vec{u}\|, \|\vec{v}\| \geq 0 \\ &\leq \|\vec{u}\|\ \|\vec{v}\| \qquad \text{since } -1 \leq \cos\theta \leq 1 \text{ implies } |\cos\theta| \leq 1. \end{aligned}$$

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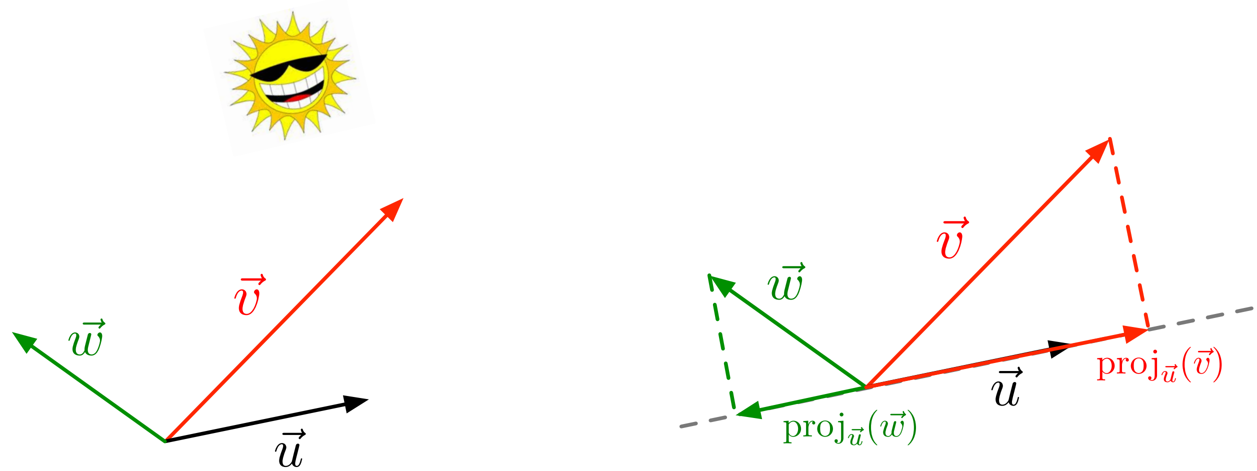
## Projections

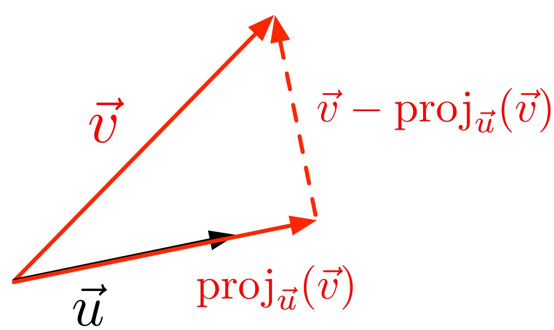
Let's look at yet another use of the dot product.

Sometimes we want to be able to determine the projection of a vector $\vec{v}$ (or $\vec{w}$) onto another vector $\vec{u}$.




Note that we are, in fact, decomposing vector $\vec{v}$ into two orthogonal parts:

$$\vec{v} \;=\; \text{proj}_{\vec{u}}(\vec{v}) \;+\; (\vec{v} - \text{proj}_{\vec{u}}(\vec{v})).$$




**Definition** If $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $\vec{u} \neq \vec{0}$, then the *projection* of $\vec{v}$ onto $\vec{u}$ is the vector $\text{proj}_{\vec{u}}(\vec{v})$ given by

$$\text{proj}_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}.$$

Let's verify that, given the definition above, $\text{proj}_{\vec{u}}(\vec{v})$ and $\vec{v} - \text{proj}_{\vec{u}}(\vec{v})$ are orthogonal.

We have

$$\begin{aligned}
\text{proj}_{\vec{u}}(\vec{v}) \cdot [\vec{v} - \text{proj}_{\vec{u}}(\vec{v})] &= \left[\left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}\right] \cdot \left[\vec{v} - \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}\right] \\
&= \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)(\vec{u}\cdot\vec{v}) - \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)^2(\vec{u}\cdot\vec{u}) \\
&= \frac{(\vec{u}\cdot\vec{v})^2}{\vec{u}\cdot\vec{u}} - \frac{(\vec{u}\cdot\vec{v})^2}{\vec{u}\cdot\vec{u}} \\
&= 0.
\end{aligned}$$

## Lines and planes

Armed with our knowledge of vectors, let us look at two familiar geometric objects: lines and planes.

**Example** Consider the line $\ell$ given by $2x - 3y = -6$.

Consider the vector $\vec{n} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

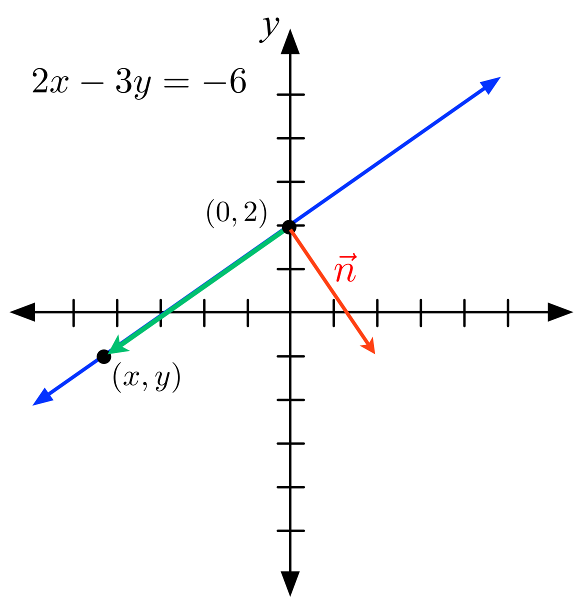
Note that this vector is orthogonal to any vector that is parallel to line $\ell$.

Pick a point on $\ell$, say $(0, 2)$, and let $(x, y)$ be an arbitrary point on $\ell$.

Then

$$\begin{bmatrix} x - 0 \\ y - 2 \end{bmatrix} = \begin{bmatrix} x \\ y - 2 \end{bmatrix}$$

is a vector along line $\ell$.



Thus, for any point $(x, y)$ on line $\ell$, we have that

$$\underbrace{\begin{bmatrix} 2 \\ -3 \end{bmatrix}}_{\text{normal vector}} \cdot \begin{bmatrix} x \\ y-2 \end{bmatrix} = 0.$$

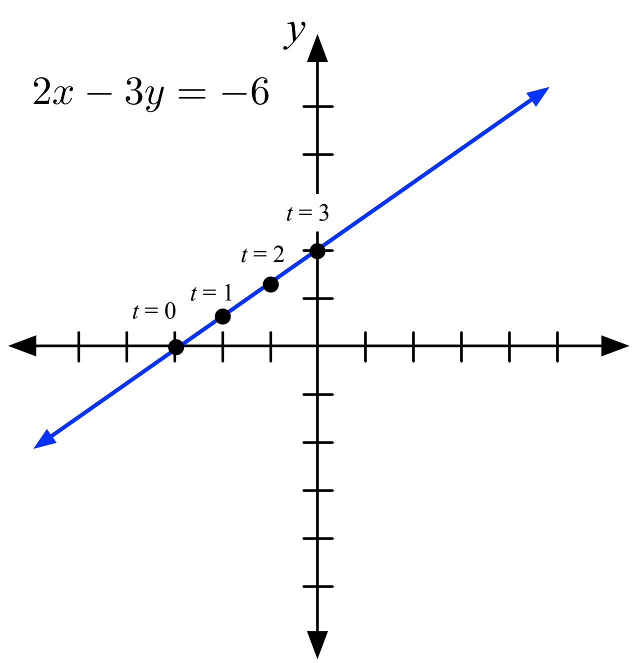
So, in general, if $\vec{n} \neq \vec{0}$ is a normal vector to a line $\ell$ in $\mathbb{R}^2$ containing point $(p_1, p_2)$, then the line has *normal form*

$$\vec{n} \cdot \begin{bmatrix} x - p_1 \\ y - p_2 \end{bmatrix} = 0 \quad \text{or} \quad \vec{n} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \vec{n} \cdot \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}.$$

However, this is not the only way to describe a line using vectors.

**Example** Consider the line from our previous example.

Imagine a particle moving along the line. Suppose at time $t = 0$ the particle is at point (-3, 0).




Also, assume the particle moves along the line such that the $x$-coordinate changes +1 unit per second.

So,

$$\text{time } t = 1 \Rightarrow \text{particle at } \left(-2, \tfrac{2}{3}\right)$$
$$\text{time } t = 2 \Rightarrow \text{particle at } \left(-1, \tfrac{4}{3}\right),$$

and more generally, if $t$ seconds have passed, the particle moves $t$ units in the $x$-direction and $\frac{2}{3}t$ units in the $y$-direction.

Then for any point $(x, y)$ on the line, we have that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix} + \begin{bmatrix} t \\ \frac{2}{3}t \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix} + t \underbrace{\begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix}}_{\text{direction vector}} .$$

Is the choice of the direction vector unique?

A line in $\mathbb{R}^2$ has *vector form*

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

where $(p_1, p_2)$ is a point on the line, $\vec{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ is a nonzero direction vector, and $t$ is a real number. The componentwise equations, given below, are called *parametric equations* of the line:

$$x = p_1 + td_1$$
$$y = p_2 + td_2.$$

Note that we can easily generalize the vector form of a line from $\mathbb{R}^2$ to $\mathbb{R}^3$.

A line in $\mathbb{R}^2$ or $\mathbb{R}^3$ has *vector form*

$$\vec{x} = \vec{p} + t\vec{d}$$

where $\vec{p}$ is a point on the line, $\vec{d} \neq \vec{0}$ is a direction vector, and $t$ is a real number. The componentwise equations are called *parametric equations* of the line.

## What about describing planes in $\mathbb{R}^3$ using vectors?

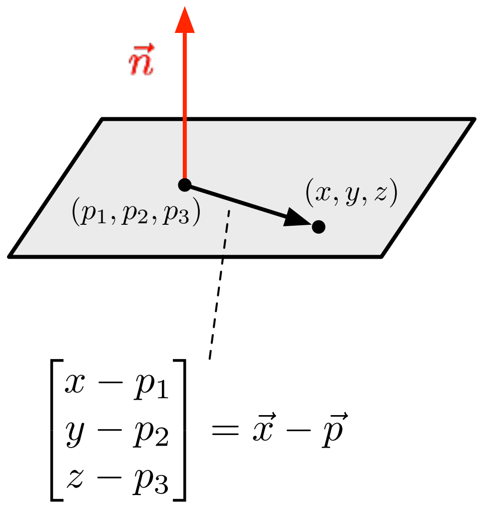
Consider a plane in $\mathbb{R}^3$.

Pick a point $(p_1, p_2, p_3)$ in the plane, and let $(x, y, z)$ be an arbitrary point in the plane. Then

$$\begin{bmatrix} x - p_1 \\ y - p_2 \\ z - p_3 \end{bmatrix}$$

is a vector in the plane for all points $(x, y, z)$ in the plane.

Using a normal vector $\vec{n}$ to the plane, the points in the plane are precisely those such that $\vec{x} - \vec{p}$ is orthogonal to $\vec{n}$.




Thus, if $\vec{n} \neq \vec{0}$ is a normal vector to a plane in $\mathbb{R}^3$ containing point $(p_1, p_2, p_3)$, then the line has *normal form*

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0 \quad \text{or} \quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}.$$

## We can also describe a plane in $\mathbb{R}^3$ with a vector form.

Key difference $\Longrightarrow$ starting from a point in the plane, we need the particle to move along two, non-parallel direction vectors rather than along only one direction vector.

A plane in $\mathbb{R}^3$ has *vector form*

$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$$

where $\vec{p}$ is a point in the plane, $\vec{u}$ and $\vec{v}$ are direction vectors (nonzero and non-parallel), and $s$ and $t$ are real numbers. The componentwise equations are called *parametric equations* of the plane.

**Important observation**

In regards to the vector or parametric forms of equations,

for a line (one-dimensional), we needed one parameter ($t$),
and for a plane (two-dimensional), we needed two parameters ($s$ and $t$).

We will talk more about dimension down the road.