Solving linear systems

Math 40, Introduction to Linear Algebra
January 2012

Breakfast cereals

Example: The box of a breakfast cereal lists the number of calories and the amounts of protein and fat contained in one serving of the cereal.

Suppose we want to prepare a mixture of these three cereals so that it contains exactly 245 calories, 6 grams of protein, and 7 grams of fat.

How do we prepare the desired mixture? (How many servings of each cereal should we combine?)

	Cheerios	Cinnamon Toast Crunch	Rice Krispies
Calories	120	130	105
Protein (g)	4	3	1
Fat (g)	2	5	2

Let

c = # of servings of Cheerios

t = # of servings of Cinnamon Toast Crunch

r = # of servings of Rice Krispies.

Then we have

$$\begin{aligned} 120c + 130t + 105r &= 245 & &\text{(calories)} \\ 4c + 3t + r &= 6 & &\text{(protein)} \\ 2c + 5t + 2r &= 7 & &\text{(fat)} \end{aligned}$$

System of linear equations

Let

c = # of servings of Cheerios

t = # of servings of Cinnamon Toast Crunch

r = # of servings of Rice Krispies.

$$\boxed{\begin{aligned} 120c + 130t + 105r &= 245 \\ 4c + 3t + r &= 6 \\ 2c + 5t + 2r &= 7 \end{aligned}}$$

system of linear equations

What makes these equations **linear**?

- all variables are raised only to first power
- each variable is multiplied only by a scalar

Def'n: A **linear equation** in the variables x_1, x_2, ..., x_n is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where a_1, a_2, ..., a_n, b are constants.

What makes this a **system** of linear equations? $\Longrightarrow$ a *finite* set of linear equations, all with the *same variables*

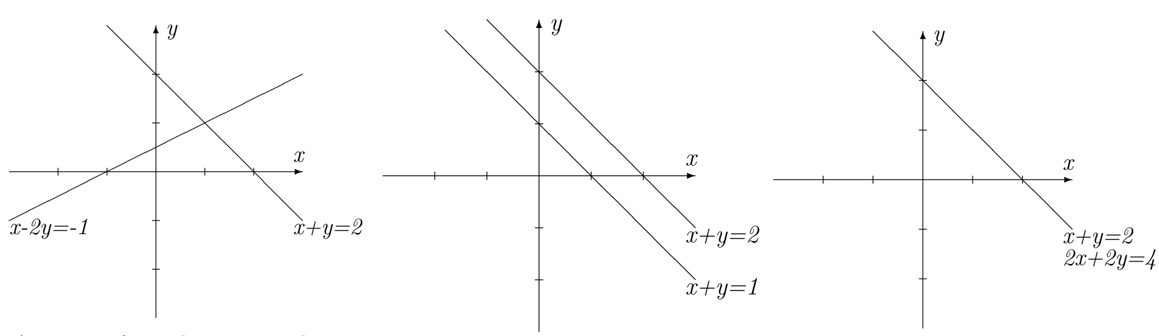
Possible sizes of solutions sets

A system of linear equations in two variables can have

- 0 solutions,
- 1 solution,
- or infinitely many solutions.

$$x + y = 2$$
$$2x + 2y = 4$$

Examples in $\mathbb{R}^2$:



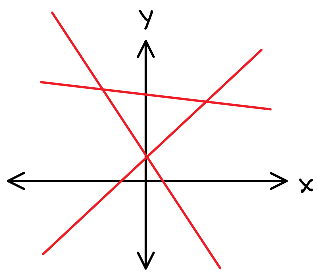
Possible sizes of solutions sets

A system of linear equations in two variables can have

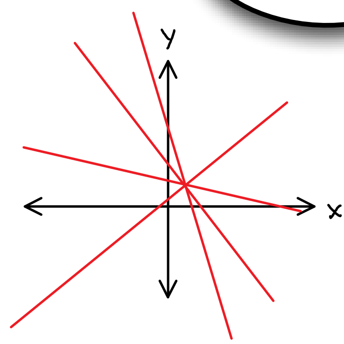
- 0 solutions,
- 1 solution,
- or infinitely many solutions.

How many solutions for a system of linear eqns. with 3 unknowns (regardless of # of eqns.)?

Other examples in $\mathbb{R}^2$:



no solution

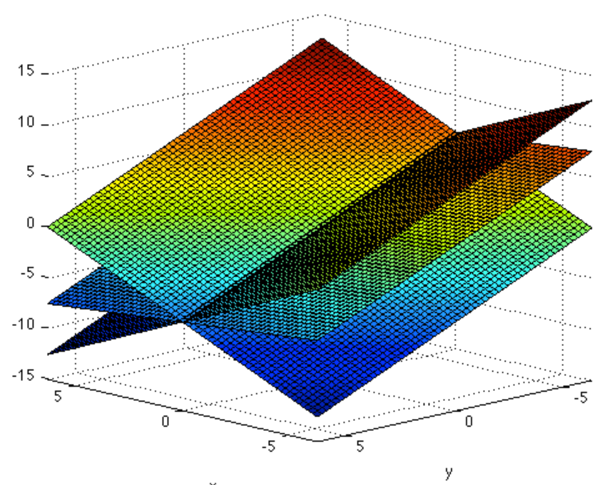


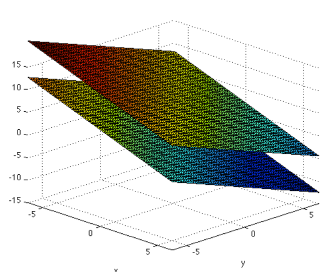
one solution

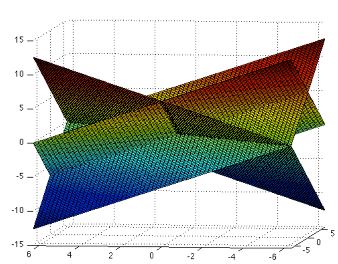
Possible sizes of solutions sets

A system of linear equations in three variables can have

- 0 solutions,
- 1 solution,
- or infinitely many solutions.



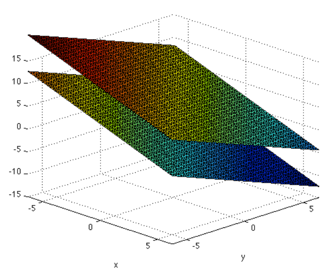
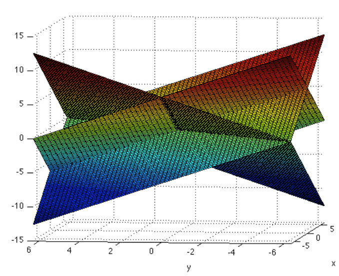
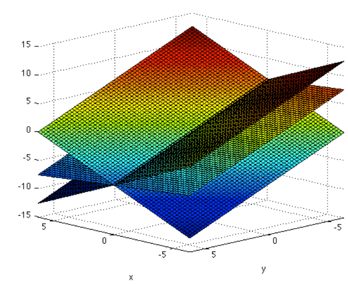




Possible sizes of solutions sets

A system of linear equations in
n variables can have

- 0 solutions,
- 1 solution,
- or infinitely many solutions.

Augmented matrix

Often, we use an **augmented matrix** to represent a linear system of equations. For now, think of a **matrix** simply as an array of numbers.

coefficients of c

Equations

$$\begin{aligned} 120c + 130t + 105r &= 245 \\ 4c + 3t + r &= 6 \\ 2c + 5t + 2r &= 7 \end{aligned}$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 120 & 130 & 105 & 245 \\ 4 & 3 & 1 & 6 \\ 2 & 5 & 2 & 7 \end{array}\right]$$

constants on RHS of eqns.

Transforming to an equivalent triangular system

Equations | Augmented matrix

Goal:

$$\begin{aligned} 120c + 130t + 105r &= 245 \\ 4c + 3t + r &= 6 \\ 2c + 5t + 2r &= 7 \end{aligned} \qquad \left[\begin{array}{ccc|c} 120 & 130 & 105 & 245 \\ 4 & 3 & 1 & 6 \\ 2 & 5 & 2 & 7 \end{array}\right]$$

eliminate c in eqns 2 and 3

swap eqn 1 and eqn 3

$$\begin{aligned} 2c + 5t + 2r &= 7 \\ 4c + 3t + r &= 6 \\ 120c + 130t + 105r &= 245 \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & 5 & 2 & 7 \\ 4 & 3 & 1 & 6 \\ 120 & 130 & 105 & 245 \end{array}\right]$$

eqn 2 - 2(eqn 1)

eqn 3 - 60(eqn 1)

$$\begin{aligned} 2c + 5t + 2r &= 7 \\ -7t - 3r &= -8 \\ -170t - 15r &= -175 \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & 5 & 2 & 7 \\ 0 & -7 & -3 & -8 \\ 0 & -170 & -15 & -175 \end{array}\right]$$

eliminate t in eqn 3

eqn 3 - $\frac{170}{7}$(eqn 2)

$$\begin{aligned} 2c + 5t + 2r &= 7 \\ -7t - 3r &= -8 \\ \tfrac{405}{7} r &= \tfrac{135}{7} \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & 5 & 2 & 7 \\ 0 & -7 & -3 & -8 \\ 0 & 0 & \frac{405}{7} & \frac{135}{7} \end{array}\right]$$

Row equivalent matrices and back substitution

$$\begin{aligned} 120c + 130t + 105r &= 245 \\ 4c + 3t + r &= 6 \\ 2c + 5t + 2r &= 7 \end{aligned} \qquad \left[\begin{array}{ccc|c} 120 & 130 & 105 & 245 \\ 4 & 3 & 1 & 6 \\ 2 & 5 & 2 & 7 \end{array}\right]$$

$\Updownarrow$ equivalent systems means same solution set

row equivalent $\Updownarrow$

$$\begin{aligned} 2c + 5t + 2r &= 7 \\ -7t - 3r &= -8 \\ \tfrac{405}{7} r &= \tfrac{135}{7} \end{aligned} \qquad \left[\begin{array}{ccc|c} 2 & 5 & 2 & 7 \\ 0 & -7 & -3 & -8 \\ 0 & 0 & \frac{405}{7} & \frac{135}{7} \end{array}\right]$$

Back substitution

$\frac{405}{7} r = \frac{135}{7} \rightarrow r = \frac{1}{3}$

$-7t - 3r = -8 \rightarrow t = 1$

$2c + 5t + 2r = 7 \rightarrow c = \frac{2}{3}$

$c = \frac{2}{3}$, $t = 1$, $r = \frac{1}{3}$

$$\boxed{\begin{bmatrix} c \\ t \\ r \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \\ \frac{1}{3} \end{bmatrix}}$$

Elementary row operations

What operations can we do to a linear system that do not change the solution set?

Elementary row operations (EROs)

- swap two rows $R_i \leftrightarrow R_j$
- multiply row by nonzero constant cR_i
- add a multiple of row to another $R_i + cR_j$

Are all of these operations reversible?

YES!

REF -- row echelon form

Row reduced matrix from cereal example:

$$\left[\begin{array}{ccc|c} 2 & 5 & 2 & 7 \\ 0 & -7 & -3 & -8 \\ 0 & 0 & \frac{405}{7} & \frac{135}{7} \end{array}\right]$$

A matrix is in row echelon form (REF) if it satisfies the following:

- any all-zero rows are at the bottom
- leading entries form a staircase pattern

first nonzero entries in each row

more formally

each leading entry is in a column to the right of the leading entry above it

Is REF of a matrix unique?

NO!

Gaussian elimination

Given a linear system, the process of

- expressing it as an augmented matrix,
- performing EROs on the augmented matrix to get it in REF,
- and, finally, using back substitution to solve the system

is called Gaussian elimination.

Can we accomplish backwards substitution via EROs done after matrix is in REF?

YES!