Solving linear systems

Math 40, Introduction to Linear Algebra
January 2012

REF -- row echelon form

Row reduced matrix
from cereal example:

$$\left[\begin{array}{ccc|c} 2 & 5 & 2 & 7 \\ 0 & -7 & -3 & -8 \\ 0 & 0 & \frac{405}{7} & \frac{135}{7} \end{array}\right]$$

A matrix is in row echelon form (REF) if it satisfies the following:

- any all-zero rows are at the bottom
- leading entries form a staircase pattern

first nonzero entries in each row

more formally

each leading entry is in a column to the right of the leading entry above it

Is REF of a matrix unique?

NO!

Gaussian elimination

Given a linear system, the process of

- expressing it as an augmented matrix,
- performing EROs on the augmented matrix to get it in REF,
- and, finally, using back substitution to solve the system

is called Gaussian elimination.

Can we accomplish backwards substitution via EROs done after matrix is in REF?

YES!

Example 1

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

rewrite linear system as augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{array}\right]$$

$$\xrightarrow[R_3 + 2R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -5 & -10 \end{array}\right]$$

matrix is in REF

We could now convert to back to equations and do back substitution...

Example 1

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matrix is in REF

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -5 & -10 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{5}R_3]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[R_2 + R_3]{R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \end{array}$$

$$\boxed{\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}}$$

RREF -- reduced row echelon form

A matrix is in reduced row echelon form (RREF) if it satisfies the following:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

- any all-zero rows are at the bottom
- leading entries form staircase pattern $\xrightarrow[\text{more formally}]{}$ each leading entry is in a column to the right of the leading entry above it
- leading entries are ones
- each column with a leading one has zeros elsewhere

Is RREF of a matrix unique?

YES!

RREF -- reduced row echelon form

A matrix is in reduced row echelon form (RREF) if it satisfies the following:

- any all-zero rows are at the bottom
- leading entries form staircase pattern
- leading entries are ones
- each column with a leading one has zeros elsewhere

Example:

$$\begin{bmatrix} 1 & \star & 0 & 0 & \star & 0 & \star & \star \\ 0 & 0 & 1 & 0 & \star & 0 & \star & \star \\ 0 & 0 & 0 & 1 & \star & 0 & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 1 & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Where must there be zeros, given leading entries above?

Gauss-Jordan elimination

Given a linear system, the process of

- expressing it as an augmented matrix,
- performing EROs on the augmented matrix to get it in RREF,
- and writing the solution to the system

is called Gauss-Jordan elimination.

In REF, RREF, or none of the above?

$$\begin{bmatrix} 1 & 0 & -9 & 0 & 4 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

REF

$$\begin{bmatrix} 1 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

REF

$$\begin{bmatrix} 1 & -\frac{1}{5} \\ 0 & 1 \end{bmatrix}$$

REF

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

none of the above

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

none of the above

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix}$$

REF

Example 2

$$\begin{aligned} 2x - 6y &= 1 \\ -x + 3y &= -1 \end{aligned}$$

$$\left[\begin{array}{cc|c} 2 & -6 & 1 \\ -1 & 3 & -1 \end{array}\right]$$

$$\xrightarrow[R_2 + \frac{1}{2}R_1]{} \left[\begin{array}{cc|c} 2 & -6 & 1 \\ 0 & 0 & -\frac{1}{2} \end{array}\right]$$

$$\cancel{\begin{aligned} 2x - 6y &= 1 \\ 0 &= -\frac{1}{2} \end{aligned}}$$

System is inconsistent

→ no solutions

Geometrically, this means the two lines are parallel.

Example 3

$$x_1 + 2x_2 + 3x_3 = 4$$
$$5x_1 + 6x_2 + 7x_3 = 8$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{array}\right]$$

$$\xrightarrow[R_2 - 5R_1]{} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{4}R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

matrix is in RREF

Example 3

$$x_1 + 2x_2 + 3x_3 = 4$$
$$5x_1 + 6x_2 + 7x_3 = 8$$

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} \textcircled{1} & 0 & -1 & -2 \\ 0 & \textcircled{1} & 2 & 3 \end{array}\right] \end{array}$$

x_1, x_2 are leading variables

x_3 is a free variable

- columns w/leading entries correspond to leading variables
- columns without leading entries correspond to free variables

$$\begin{array}{r} x_1 - x_3 = -2 \\ x_2 + 2x_3 = 3 \\ x_3 \text{ is free} \end{array} \longrightarrow \begin{array}{l} x_1 = x_3 - 2 \\ x_2 = -2x_3 + 3 \\ x_3 \text{ is free} \end{array}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 - 2 \\ -2x_3 + 3 \\ x_3 \end{bmatrix} = \boxed{x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} \quad \text{where } x_3 \text{ is any real number}}$$

system has infinite
# of solutions

line in $\mathbb{R}^3$

Example 4

$$\begin{aligned} -2x_1 + x_2 + 4x_3 &= 4 \\ 3x_1 - \frac{3}{2}x_2 - 8x_3 &= -12 \\ 2x_1 - x_2 - 6x_3 &= -10 \end{aligned}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 4 & 4 \\ 3 & -\frac{3}{2} & -8 & -12 \\ 2 & -1 & -6 & -10 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

RREF

x_1, x_3 are leading variables

x_2 is a free variable

$$\begin{aligned} x_1 &= \frac{1}{2}x_2 + 4 \\ x_2 &\text{ is free} \\ x_3 &= 3 \end{aligned} \longrightarrow \vec{x} = \begin{bmatrix} \frac{1}{2}x_2 + 4 \\ x_2 \\ 3 \end{bmatrix} = \boxed{x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \quad \text{for any } x_2 \in \mathbb{R}}$$

Recognizing the size of solution set

Suppose augmented matrix is in RREF.

represents a nonzero value

(1) If any row is of the form $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & \star \end{array}\right]$ then the system is inconsistent and there are **no solutions**.

(2) If the system is consistent and if there exist free variables, then there are an **infinite number of solutions**.

(3) Otherwise, system has a **unique solution**.

What does a row of all zeros in RREF mean?