Homogeneous systems and rank

Math 40, Introduction to Linear Algebra
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Special case: homogeneous system

Homogeneous system

$$\left[\begin{array}{ccc|c} & & & 0 \\ & & & 0 \\ & & & \vdots \\ & & & 0 \end{array}\right]$$

constants in all equations are zero

- Any homogeneous system is consistent since $\vec{x} = \vec{0}$ is always a solution.

- If homogeneous system has *m* equations and *n* variables with $m < n$, then system has infinite # of solutions.

Do we need an augmented matrix when performing Gauss-Jordan elimination on a homogeneous system?

Rank of a matrix

rank of a matrix = # of nonzero rows in its RREF (or REF)

- can think of rank as the # of "independent" rows of the matrix
- rank of the matrix associated with a linear system is the # of non-redundant equations in the system
- equivalently, the rank is the # of leading ones in the RREF of the matrix

$$\begin{aligned} x + y + z - w &= 0 \\ 2x + 2y + z - 3w &= 0 \\ -x - y + z + 3w &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 2 & 1 & -3 \\ -1 & -1 & 1 & 3 \end{bmatrix} \xrightarrow{\text{EROs}} \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

eqn 3 = 3(eqn 1) - 2(eqn 2)

rank is 2

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Theorem: Let A be the coefficient matrix of a linear system with n variables. If the system is consistent, then

# of free variables = n - rank(A).

Spanning sets and linear independence

Introduction

We begin with the notion of a linear combination.

Definition A *linear combination* of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is any vector of the form

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_k\vec{v}_k,$$

where $c_1, c_2, \ldots, c_k$ are scalars.

Example Consider

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In this case, we say that $\left[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right]$ is a linear combination of $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$.

Can we represent $\left[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right]$ by other linear combinations besides the one given above?

$$\text{YES} \quad \Longrightarrow \quad \text{e.g.,} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 5 \\ 6 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ 3 \end{bmatrix} + (-4)\begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}.$$

Motivating example

Can we express any vector in $\mathbb{R}^2$ as a linear combination of

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}?$$

fundamental building blocks of $\mathbb{R}^2$

"basis" of $\mathbb{R}^2$

YES! Any $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ can be expressed as

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

What about $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as fundamental building blocks of $\mathbb{R}^2$?

Two important qualities of $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$:

- enough vectors in S to express every vector in $\mathbb{R}^2$ as a linear combination of those in S } *span*
- no vectors in S are redundant, i.e., vectors in S are "independent" (no vector in S can be expressed as a linear combination of the others) } *linear independence*

Span

Example Is $\begin{bmatrix} -3 \\ 8 \\ -5 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$?

In other words, are there scalars c_1, c_2 such that

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ -5 \end{bmatrix}?$$

This is a linear system! (two unknowns $\Rightarrow$ c_1, c_2)

$$\left[\begin{array}{cc|c} 1 & 3 & -3 \\ 2 & -1 & 8 \\ -1 & 1 & -5 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

augmented matrix

Thus, the solution is $c_1 = 3, c_2 = -2$, which implies that

$$\underbrace{\begin{bmatrix} -3 \\ 8 \\ -5 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}}_{\text{YES, it is a linear combination.}}$$

We say that $\begin{bmatrix} -3 \\ 8 \\ -5 \end{bmatrix}$ is in the *span* of $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$.

Definition The *span* of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$ is

$$\begin{aligned} \operatorname{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k) &= \text{set of all linear combinations of } \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \\ &= \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k : c_1, \ldots, c_k \in \mathbb{R}\}. \end{aligned}$$

Remarks

- $\operatorname{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$ is a **set of vectors** (the set of all vectors that can be built from $\vec{v}_1, \ldots, \vec{v}_k$)
- If $S = \{\vec{v}_1, \ldots, \vec{v}_k\}$, then $\operatorname{span}(S) = \operatorname{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$.
- If $\operatorname{span}(S) = \mathbb{R}^n$, i.e., every vector in $\mathbb{R}^n$ is a linear combination of the vectors in S, then we say S is a *spanning set* of $\mathbb{R}^n$.

Example Is $\left\{ \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix} \right\}$ a spanning set of $\mathbb{R}^3$?

Equivalently, for any $\vec{b} \in \mathbb{R}^3$, can $\vec{b}$ be expressed as a linear combination of the three given vectors, i.e, do there exist scalars c_1, c_2, c_3 such that

$$c_1 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}?$$

$$\text{Is } \left[\begin{array}{ccc|c} -1 & 2 & 4 & b_1 \\ 2 & 1 & 7 & b_2 \\ 3 & -1 & 3 & b_3 \end{array}\right] \begin{array}{c} \text{consistent for all choices} \\ \text{of } b_1, b_2, b_3 \in \mathbb{R} \end{array} \;?$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 4 & b_1 \\ 2 & 1 & 7 & b_2 \\ 3 & -1 & 3 & b_3 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{ccc|c} 1 & -2 & -4 & -b_1 \\ 0 & 5 & 15 & b_2 + 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 - b_2 \end{array}\right]$$

Because $b_3 + b_1 - b_2 \neq 0$ for all $b_1, b_2, b_3 \in \mathbb{R}$, the given set of vectors is NOT a spanning set of $\mathbb{R}^3$.

Summary Given $\vec{v}_1, \ldots, \vec{v}_k, \vec{b} \in \mathbb{R}^n$,

$\vec{b} \in \text{span}(\vec{v}_1, \ldots, \vec{v}_k) \iff \vec{b}$ is a linear comb. of $\vec{v}_1, \ldots, \vec{v}_k \iff [\,A \mid \vec{b}\,]$ has a solution, where A is matrix w/columns as $\vec{v}_1, \ldots, \vec{v}_k$

Given $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$,

$S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is a spanning set of $\mathbb{R}^n \iff \text{span}(S) = \mathbb{R}^n \iff [\,A \mid \vec{b}\,]$ has a solution <u>for any $\vec{b} \in \mathbb{R}^n$</u>

Linear independence

Example Recall we previously found that

$$\begin{bmatrix} -3 \\ 8 \\ 5 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

Rearranging terms, we have

$$3 \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}_{\vec{v}_1} - 2 \underbrace{\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}}_{\vec{v}_2} - \underbrace{\begin{bmatrix} -3 \\ 8 \\ 5 \end{bmatrix}}_{\vec{v}_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, we have a nontrivial way to express $\vec{0}$ as a linear combination of v_1, v_2, v_3. This is the definition of *linear dependence*.

Definition We say vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are *linearly dependent* if $\exists$ scalars $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}. \quad (\star)$$

Otherwise, the vectors are *linearly independent*, which means the only solution to $(\star)$ is the trivial solution $c_1 = \cdots = c_k = 0$.