Spanning sets and linear independence

Span (continued)

Summary

Given $\vec{v}_1, \ldots, \vec{v}_k, \vec{b} \in \mathbb{R}^n$,

$\vec{b} \in \text{span}(\vec{v}_1, \ldots, \vec{v}_k) \iff$ $\vec{b}$ is a linear comb. of $\vec{v}_1, \ldots, \vec{v}_k$ $\iff$ $[\, A \mid \vec{b}\,]$ has a solution, where A is matrix w/columns as $\vec{v}_1, \ldots, \vec{v}_k$

Given $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$,

$S = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is a spanning set of $\mathbb{R}^n$ $\iff$ $\text{span}(S) = \mathbb{R}^n$ $\iff$ $[\, A \mid \vec{b}\,]$ has a solution for any $\vec{b} \in \mathbb{R}^n$

Linear independence

Example Recall we previously found that

$$\begin{bmatrix} -3 \\ 8 \\ -5 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

Rearranging terms, we have

$$3 \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}_{\vec{v}_1} - 2 \underbrace{\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}}_{\vec{v}_2} - \underbrace{\begin{bmatrix} -3 \\ 8 \\ -5 \end{bmatrix}}_{\vec{v}_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus, we have a nontrivial way to express $\vec{0}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$. This is the definition of *linear dependence*.

Definition We say vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are *linearly dependent* if $\exists$ scalars $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}. \quad (\star)$$

Otherwise, the vectors are *linearly independent*, which means the only solution to $(\star)$ is the trivial solution $c_1 = \cdots = c_k = 0$.

To determine if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent or not, we need to know if $\exists$ nontrivial solution to

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0} \quad \Longrightarrow \quad \underbrace{\left[\begin{array}{cccc|c} | & | & & | & 0 \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k & \vdots \\ | & | & & | & 0 \end{array}\right]}_{\text{augmented matrix}}$$

This is a homogeneous linear system! We need to determine if the system has one solution or more than one solution.

$$\vec{v}_1, \ldots, \vec{v}_k \text{ are linearly independent} \iff [\,A \mid \vec{0}\,] \text{ has a unique solution, namely } \vec{0} \text{ (where } A \text{ has } \vec{v}_1, \ldots, \vec{v}_k \text{ as columns)}$$

Example Are the following vectors linearly independent?

$$\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$$

We do EROs on the appropriate augmented matrix to get

$$\left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{array}\right].$$

We see that there are no free variables (i.e., every column has a leading entry), so the system has a unique solution, $\vec{0}$. Thus, the vectors are

linearly independent.

Theorem.

Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are linearly dependent *if and only if* *one of the vectors can be written as a linear combination of the others.*

Facts Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of vectors in $\mathbb{R}^n$.

(1) If $\vec{v}_i = \vec{0}$ for some i, then S is linearly dependent.

(2) If one vector in S is a linear combination of the other vectors in S, then S is linearly dependent.

(3) If $k > n$ (more vectors than components), then S is linearly dependent.

Homogeneous system w/more variables than equations must have a free variable.

Example Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -10 \\ 21 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{7} \\ 7 \end{bmatrix} \right\}$ linearly independent?

NO! The vectors are linearly dependent since we have 4 vectors in $\mathbb{R}^3$. (See fact 3 above.)

Matrices

At its core, linear algebra is the study of linear transformations and their algebraic properties. We'll see, down the road, that there is an intimate relationship between a linear transformation and a matrix.

Recall...

Definition A *matrix* is a rectangular array of numbers.

Example $A = \begin{bmatrix} 1 & 2 \\ -\frac{3}{2} & 3 \\ 0 & -5 \end{bmatrix}$ is a 3×2 matrix.

a_{ij} denotes the entry of A in row i and column j, so, for example, $a_{12} = 2$ and $a_{21} = -\frac{3}{2}$.

Definition If A is an $n \times n$ matrix (i.e., # of rows = # of cols.), then we say that A is a *square matrix*.

Matrix operations

- **Equality:**

$$A = B \iff A, B \text{ are same size and } a_{ij} = b_{ij} \ \forall\, i, j$$

- **Matrix addition:** A, B are $m \times n$ matrices

$$C = A + B \text{ is the } m \times n \text{ matrix defined as } c_{ij} = a_{ij} + b_{ij} \ \forall\, i, j$$

add entrywise

- **Scalar multiplication:** $m \times n$ matrix A, scalar c

$$cA \text{ is the } m \times n \text{ matrix with entries } ca_{ij} \ \forall\, i, j$$

Example

$$2\begin{bmatrix} 2 & \frac{5}{2} & 1 \\ -1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 1 & -4 \\ 1 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 & -2 \\ -1 & 5 & 6 \end{bmatrix}$$

Remark The set of all $m \times n$ matrices with real entries (denoted $\mathbb{R}^{m \times n}$ or $M_{m \times n}(\mathbb{R})$) with the operations of matrix addition and scalar multiplication form a vector space.

A, B, C are $m \times n$ matrices, c, d are scalars

(1)	$A + B$ is an $m \times n$ matrix	(closure under addition)
(2)	$A + B = B + A$	(commutativity)
(3)	$(A + B) + C = A + (B + C)$	(associativity)
(4)	$A + 0 = A$	(existence of additive identity)
(5)	$A + (-A) = 0$	(existence of additive inverses)
(6)	cA is an $m \times n$ matrix	(closure under scalar multiplication)
(7)	$c(A + B) = cA + cB$	(distributivity)
(8)	$(c + d)A = cA + dA$	(distributivity)
(9)	$c(dA) = (cd)A$	
(10)	$1A = A$	

Matrix multiplication $\Longrightarrow$ see slides

Matrix multiplication

Math 40, Introduction to Linear Algebra
Monday, January 30, 2012

Matrix-vector multiplication: two views

- 1st perspective: $A\vec{x}$ is linear combination of columns of A

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}}_{\vec{x}} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Matrix-vector multiplication: two views

- 1st perspective: $A\vec{x}$ is linear combination of columns of A

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}}_{\vec{x}} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 21 \end{bmatrix}}$$

- 2nd perspective: $A\vec{x}$ is computed as dot product of rows of A with vector $\vec{x}$

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} \text{dot product of } \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \\ \\ \end{bmatrix}$$

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$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} 4 \\ \text{dot product of } \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \text{ and } \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 21 \end{bmatrix}}$$

Notice that **# of columns of A = # of rows of $\vec{x}$**.
This is a requirement in order for matrix multiplication to be defined.

Matrix multiplication (in general)

$$\underset{A}{\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}} \underset{B}{\begin{bmatrix} 4 & 2 & 1 \\ 3 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}}$$

Note that
$\#$ cols. of A $=$ $\#$ of rows of B

$$AB = \underset{m \text{ x } n}{\begin{bmatrix} & A & \end{bmatrix}} \underset{n \text{ x } p}{\begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \\ | & | & & | \end{bmatrix}} = \underset{m \text{ x } p}{\begin{bmatrix} | & | & & | \\ A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \\ | & | & & | \end{bmatrix}}$$

Each column of AB is a linear combination of columns of A.

Matrix multiplication (in general)

$$\underset{A}{\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}} \underset{B}{\begin{bmatrix} 4 & 2 & 1 \\ 3 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}} = \begin{bmatrix} 4 & 5 & 6 \\ 21 & 9 & 19 \end{bmatrix}$$

Note that
$\#$ cols. of A $=$ $\#$ of rows of B

Computing AB via linear combinations of columns of A:

$$\text{1st column of } AB = 4\begin{bmatrix}1\\2\end{bmatrix} + 3\begin{bmatrix}-2\\1\end{bmatrix} + 2\begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}4\\21\end{bmatrix}$$

$$\text{2nd column of } AB = 2\begin{bmatrix}1\\2\end{bmatrix} + 0\begin{bmatrix}-2\\1\end{bmatrix} + 1\begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}5\\9\end{bmatrix}$$

$$\text{3rd column of } AB = 1\begin{bmatrix}1\\2\end{bmatrix} + 2\begin{bmatrix}-2\\1\end{bmatrix} + 3\begin{bmatrix}3\\5\end{bmatrix} = \begin{bmatrix}6\\19\end{bmatrix}$$

While you should understand this approach, it is often easier to multiply matrices via dot products.

Matrix multiplication (in general)

In terms of dot products,

$$\text{the } (i,j)\text{-entry of } AB = [i\text{th row of } A] \cdot [j\text{th column of } B]$$

viewed as column vectors

$$\underset{A}{\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}} \underset{B}{\begin{bmatrix} 4 & 2 & 1 \\ 3 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}} = \begin{bmatrix} 4 & & \\ & & \end{bmatrix}$$

since $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = 4$

Matrix multiplication (in general)

In terms of dot products,

$$\text{the } (i,j)\text{-entry of } AB = [i\text{th row of } A] \cdot [j\text{th column of } B]$$

viewed as column vectors

$$\underset{A}{\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}} \underset{B}{\begin{bmatrix} 4 & 2 & 1 \\ 3 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}} = \begin{bmatrix} 4 & 5 & 6 \\ 21 & 9 & 19 \end{bmatrix}$$

since $\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 19$

Matrix multiplication

What sizes of matrices can be multiplied together?

For m x n matrix A and n x p matrix B, the matrix product AB is an m x p matrix.

$$m \times n \quad n \times p$$

"inner" parameters must match

"outer" parameters become parameters of matrix AB

If A is a square matrix and k is a positive integer, we define

$$A^k = \underbrace{A \cdot A \cdots A}_{k \text{ factors}}$$