Matrices, transposes, and inverses

Math 40, Introduction to Linear Algebra
Wednesday, February 1, 2012

Matrix-vector multiplication: two views

- 1st perspective: $A\vec{x}$ is linear combination of columns of A

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}}_{\vec{x}} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 21 \end{bmatrix}}$$

- 2nd perspective: $A\vec{x}$ is computed as dot product of rows of A with vector $\vec{x}$

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} 4 \\ \text{dot product of } \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \text{ and } \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 21 \end{bmatrix}}$$

Notice that **# of columns of A = # of rows of $\vec{x}$** .
This is a requirement in order for matrix multiplication to be defined.

Matrix multiplication

What sizes of matrices can be multiplied together?

For m x n matrix A and n x p matrix B, the matrix product AB is an m x p matrix.

m x n n x p

"inner" parameters must match

"outer" parameters become parameters of matrix AB

If A is a square matrix and k is a positive integer, we define

$$A^k = \underbrace{A \cdot A \cdots A}_{k \text{ factors}}$$

Properties of matrix multiplication

Most of the properties that we expect to hold for matrix multiplication do....

$$A(B + C) = AB + AC$$

$$(AB)C = A(BC)$$

$$k(AB) = (kA)B = A(kB) \text{ for scalar } k$$

.... except commutativity!!

In general, $AB \neq BA$.

Matrix multiplication not commutative

Problems with hoping AB and BA are equal:

In general, $AB \neq BA$.

- BA may not be well-defined.

  (e.g., A is 2 x 3 matrix, B is 3 x 5 matrix)

- Even if AB and BA are both defined, AB and BA may not be the same size.

  (e.g., A is 2 x 3 matrix, B is 3 x 2 matrix)

- Even if AB and BA are both defined and of the same size, they still may not be equal.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \neq \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Truth or fiction?

Question 1 For n x n matrices A and B, is

$$A^2 - B^2 = (A - B)(A + B) \ ?$$

No!!

$$(A - B)(A + B) = A^2 + \underbrace{AB - BA}_{\neq 0} - B^2$$

Question 2 For n x n matrices A and B, is $(AB)^2 = A^2B^2$?

No!!

$$(AB)^2 = ABAB \neq AABB = A^2B^2$$

Matrix transpose

Definition The ***transpose*** of an m x n matrix A is the n x m matrix A^T obtained by interchanging rows and columns of A,

$$\text{i.e., } (A^T)_{ij} = A_{ji} \quad \forall\, i, j.$$

Example

$$A = \begin{bmatrix} 1 & 3 & 5 & -2 \\ 5 & 3 & 2 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 5 \\ 3 & 3 \\ 5 & 2 \\ -2 & 1 \end{bmatrix}$$

Transpose operation can be viewed as flipping entries about the diagonal.

Definition A square matrix A is ***symmetric*** if $A^T = A$.

Properties of transpose

(1) $(A^T)^T = A$

apply twice -- get back to where you started

(2) $(A+B)^T = A^T + B^T$

(3) For a scalar c, $(cA)^T = cA^T$

(4) $(AB)^T = B^T A^T$

To prove this, we show that

$$[(AB)^T]_{ij} = \\ \vdots \\ = [(B^T A^T)]_{ij}$$

Exercise

Prove that for any matrix A, $A^T A$ is symmetric.

Special matrices

Definition A matrix with all zero entries is called a ***zero matrix*** and is denoted 0.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Definition A square matrix is ***upper-triangular*** if all entries below main diagonal are zero.

analogous definition for a ***lower-triangular*** matrix

$$A = \begin{bmatrix} 2 & \frac{1}{4} & 5 \\ 0 & 6 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Definition A square matrix whose off-diagonal entries are all zero is called a ***diagonal matrix***.

$$A = \begin{bmatrix} -\frac{3}{8} & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Definition The ***identity matrix***, denoted I_n, is the n x n diagonal matrix with all ones on the diagonal.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity matrix

Definition The ***identity matrix***, denoted I_n, is the n x n diagonal matrix with all ones on the diagonal.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Important property of identity matrix

If A is an m x n matrix, then

$$I_m A = A \quad \text{and} \quad AI_n = A.$$

If A is a square matrix, then

$$IA = A = AI.$$

The notion of inverse

Exploration Consider the set of real numbers, and say that we have the equation

$$3x = 2$$

and we want to solve for x. What do we do?

We multiply both sides of the equation by $\frac{1}{3}$ to obtain

$$\frac{1}{3}(3x) = \frac{1}{3}(2) \implies x = \frac{2}{3}.$$

multiplicative inverse of 3 since $\frac{1}{3}(3) = 1$

Now, consider the linear system

$$\begin{aligned} 3x_1 - 5x_2 &= 6 \\ -2x_1 + 3x_2 &= -1 \end{aligned}$$

Notice that we can rewrite equations as

$$\underbrace{\begin{bmatrix} 3 & -5 \\ -2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 6 \\ -1 \end{bmatrix}}_{\vec{b}}$$

How do we isolate the vector $\vec{x}$ by itself on LHS?

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$$\underbrace{\begin{bmatrix} \quad ? \quad \end{bmatrix} \left(\begin{bmatrix} 3 & -5 \\ -2 & 3 \end{bmatrix}\right.}_{\text{want this equal to identity matrix, } I} \left.\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} \quad ? \quad \end{bmatrix} \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} -13 \\ -9 \end{bmatrix}}$$

Matrix inverses

Definition A square matrix A is *invertible* (or *nonsingular*) if $\exists$ matrix B such that $AB = I$ and $BA = I$. (We say B is an *inverse* of A.)

Example

$$A = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} \text{ is invertible because for } B = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix},$$

$$\text{we have } AB = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\text{and likewise } BA = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

The notion of an inverse matrix only applies to square matrices.

- For rectangular matrices of full rank, there are one-sided inverses.
- For matrices in general, there are pseudoinverses, which are a generalization to matrix inverses.

Example Find the inverse of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. We have

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Longrightarrow \quad a + c = 1 \text{ and } a + c = 0 \qquad \text{IMPOSSIBLE!}$$

Therefore, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible (or singular).

Take-home message: Not all square matrices are invertible.

Important questions:

- When does a square matrix have an inverse?
- If it does have an inverse, how do we compute it?
- Can a matrix have more than one inverse?

Theorem. *If A is invertible, then its inverse is unique.*

Proof. Assume A is invertible. Suppose, by way of contradiction, that the inverse of A is not unique, i.e., let B and C be two distinct inverses of A. Then, by def'n of inverse, we have

$$\begin{aligned} BA = I = AB \quad (1) \\ \text{and } CA = I = AC. \quad (2) \end{aligned}$$

It follows that

$$\begin{aligned} B &= BI && \text{by def'n of identity matrix} \\ &= B(AC) && \text{by (2) above} \\ &= (BA)C && \text{by associativity of matrix mult.} \\ &= IC && \text{by (1) above} \\ &= C. && \text{by def'n of identity matrix} \end{aligned}$$

Thus, $B = C$, which contradicts the previous assumption that $B \neq C$. $\Rightarrow\Leftarrow$ So it must be that case that the inverse of A is unique. ■

Take-home message: The inverse of a matrix A is unique, and we denote it A^{-1}.

Theorem (Properties of matrix inverse)**.**

(a) If A is invertible, then A^{-1} is itself invertible and $(A^{-1})^{-1} = A$.

(b) If A is invertible and $c \neq 0$ is a scalar, then cA is invertible and $(cA)^{-1} = \frac{1}{c}A^{-1}$.

(c) If A and B are both $n \times n$ invertible matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

"socks and shoes rule" – similar to transpose of AB

generalization to product of n matrices

(d) If A is invertible, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

To prove (d), we need to show that there is some matrix ___ such that

$$\_\_\_ \, A^T = I \quad \text{and} \quad A^T \, \_\_\_ = I.$$

Proof of (d). Assume A is invertible. Then A^{-1} exists and we have

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

and

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

So A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$. ■

Question: If A and B are invertible $n \times n$ matrices, what can we say about $A + B$?

There is no guarantee $A + B$ is invertible even if A and B themselves are invertible! In other words, we CANNOT say that $(A + B)^{-1} = A^{-1} + B^{-1}$.

How do we compute the inverse of a matrix, if it exists?

Inverse of a 2×2 matrix: Consider the special case where A is a 2×2 matrix with $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. If $ad - bc \neq 0$, then A is invertible and its inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$\star$ Exercise: Check that $AA^{-1} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] = A^{-1}A$.

Example For $A = \begin{bmatrix} -2 & 1 \\ 3 & -3 \end{bmatrix}$, we have

$$A^{-1} = \frac{1}{3} \begin{bmatrix} -3 & -1 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{3} \\ -1 & -\frac{2}{3} \end{bmatrix}.$$

We can easily check that

$$AA^{-1} = \begin{bmatrix} -2 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} -1 & -\frac{1}{3} \\ -1 & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A^{-1}A = \begin{bmatrix} -1 & -\frac{1}{3} \\ -1 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

How do we find inverses of matrices that are larger than 2×2 matrices?

Theorem. *If some EROs reduce a square matrix A to the identity matrix I, then the same EROs transform I to A^{-1}.*

$$\left[\begin{array}{c|c} A & I \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

If we can transform A into I, then we will obtain A^{-1}. If we cannot do so, then A is not invertible.

Example: Find the inverse of the matrix $A = \begin{bmatrix} -1 & -3 & 1 \\ 3 & 6 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} -1 & -3 & 1 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2+3R_1} \left[\begin{array}{ccc|ccc} -1 & -3 & 1 & 1 & 0 & 0 \\ 0 & -3 & 3 & 3 & 1 & 0 \\ 0 & -3 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3-R_2]{-R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -3 & 3 & 3 & 1 & 0 \\ 0 & 0 & -1 & -2 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[-R_3]{R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & -3 & 3 & 3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & 1 & -1 & -1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[R_2+R_3]{R_1-2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 2 \\ 0 & 1 & 0 & 1 & \frac{2}{3} & -1 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right]$$

Thus, A is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -1 & 2 \\ 1 & \frac{2}{3} & -1 \\ 2 & 1 & -1 \end{bmatrix}.$$

Why does this work? $\implies$ discussion next class