Matrix inverses

Recall...

Definition A square matrix A is *invertible* (or *nonsingular*) if $\exists$ matrix B such that $AB = I$ and $BA = I$. (We say B is an *inverse* of A.)

Remark Not all square matrices are invertible.

Theorem. *If A is invertible, then its inverse is unique.*

Remark When A is invertible, we denote its inverse as A^{-1}.

Theorem. *If A is an $n \times n$ invertible matrix, then the system of linear equations given by $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.*

Proof. Assume A is an invertible matrix. Then we have

$$A(A^{-1}\vec{b}) \overset{\text{by associativity of matrix mult.}}{=} (AA^{-1})\vec{b} \underset{\text{by def'n of inverse}}{=} I\vec{b} \overset{\text{by def'n of identity}}{=} \vec{b}.$$

Thus, $\vec{x} = A^{-1}\vec{b}$ is a solution to $A\vec{x} = \vec{b}$.

Suppose $\vec{y}$ is another solution to the linear system. It follows that $A\vec{y} = \vec{b}$, but multiplying both sides by A^{-1} gives $\vec{y} = A^{-1}\vec{b} = \vec{x}$. ■

Theorem (Properties of matrix inverse)**.**

(a) If A is invertible, then A^{-1} is itself invertible and $(A^{-1})^{-1} = A$.

(b) If A is invertible and $c \neq 0$ is a scalar, then cA is invertible and $(cA)^{-1} = \frac{1}{c}A^{-1}$.

(c) If A and B are both $n \times n$ invertible matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

"socks and shoes rule" – similar to transpose of AB
generalization to product of n matrices

(d) If A is invertible, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

To prove (d), we need to show that the matrix B that satisfies $BA^T = I$ and $A^T B = I$ is $B = (A^{-1})^T$.

Proof of (d). Assume A is invertible. Then A^{-1} exists and we have

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$$

and

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I.$$

So A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$. ■

Recall...

How do we compute the inverse of a matrix, if it exists?

Inverse of a 2×2 matrix: Consider the special case where A is a 2×2 matrix with $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. If $ad - bc \neq 0$, then A is invertible and its inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

How do we find inverses of matrices that are larger than 2×2 matrices?

Theorem. *If some EROs reduce a square matrix A to the identity matrix I, then the same EROs transform I to A^{-1}.*

$$\left[\begin{array}{c|c} A & I \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

If we can transform A into I, then we will obtain A^{-1}. If we cannot do so, then A is not invertible.

Can we capture the effect of an ERO through matrix multiplication?

Definition An *elementary matrix* is any matrix obtained by doing an ERO on the identity matrix.

Examples

$$\underset{\text{on } 4\times 4 \text{ identity}}{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \underset{\text{on } 3\times 3 \text{ identity}}{R_1 - 4R_3} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that

$$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} - 4a_{31} & a_{12} - 4a_{32} & a_{13} - 4a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Left mult. of A by row vector is a linear comb. of rows of A.

Remark An elementary matrix E is invertible and E^{-1} is elementary matrix corresponding to the "reverse" ERO of one associated with E.

Example If E is 2nd elementary matrix above, then "reverse" ERO is $R_1 + 4R_3$ and $E^{-1} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Remark When finding A^{-1} using Gauss-Jordan elimination of $[\ A \mid I\ ]$, if we keep track of EROs, and if $E_1, E_2, \ldots, E_k$ are corresponding elem. matrices, then we have

$$E_k E_{k-1} \cdots E_1 A = I \quad \Longrightarrow \quad A = E_1^{-1} \cdots E_{k-1}^{-1} E_k^{-1}.$$

Theorem (Fundamental Thm of Invertible Matrices)**.**
For an $n \times n$ matrix, the following are equivalent:

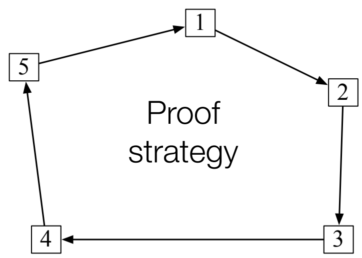
(1) A is invertible.

(2) $A\vec{x} = \vec{b}$ has a unique solution for any $\vec{b} \in \mathbb{R}^n$.

(3) $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = 0$.

(4) The RREF of A is I.

(5) A is product of elementary matrices.



Proof.

(1) $\Rightarrow$ (2):
Proven in first theorem of today's lecture

(2) $\Rightarrow$ (3):
If $A\vec{x} = \vec{b}$ has unique sol'n for any $\vec{b} \in \mathbb{R}^n$, then in particular, $A\vec{x} = \vec{0}$ has a unique sol'n. Since $\vec{x} = \vec{0}$ is a solution to $A\vec{x} = \vec{0}$, it must be the unique one.

(3) $\Rightarrow$ (4):
If $A\vec{x} = \vec{0}$ has unique sol'n $\vec{x} = 0$, then augmented matrix has no free variables and a leading one in every column:

$$\left[\begin{array}{cccc|c} 1 & & & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & & & 1 & 0 \end{array}\right]$$

so RREF of A is I.

(4) $\Rightarrow$ (5):
$E_k \cdots E_1 A = \text{RREF of } A = I$
and elem. matrices are invertible
$\implies \quad A = E_1^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$.

(5) $\Rightarrow$ (1):
Since $A = E_k \cdots E_1$ and E_i invertible $\forall\ i$, A is product of invertible matrices so it is itself invertible. ∎

Theorem. *Let A be a square matrix. If B is a square matrix such that either $AB = I$ or $BA = I$, then A is invertible and $B = A^{-1}$.*

Proof. Suppose A, B are $n \times n$ matrices and that $BA = I$. Then consider the homogeneous system $A\vec{x} = \vec{0}$. We have

$$B(A\vec{x}) = B\vec{0} \quad \Longrightarrow \quad \underbrace{(BA)}_{I}\vec{x} = \vec{0} \quad \Longrightarrow \quad \vec{x} = \vec{0}.$$

Since $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$, by the Fundamental Thm of Inverses, we have that A is invertible, i.e., A^{-1} exists. Thus,

$$(BA)A^{-1} = IA^{-1} \quad \Longrightarrow \quad B\underbrace{(AA^{-1})}_{I} = A^{-1} \quad \Longrightarrow \quad B = A^{-1}.$$

We leave the case of $AB = I$ as an exercise. ■

Definition The vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \in \mathbb{R}^n$, where $\vec{e}_i$ has a one in its ith component and zeros elsewhere, are called *standard unit vectors*.

Example The 4×4 identity matrix can be expressed as

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} | & | & | & | \\ \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \\ | & | & | & | \end{bmatrix}$$

Theorem. *If some EROs reduce a square matrix A to the identity matrix I, then the same EROs transform I to A^{-1}.*

Why does this work?

Want to solve $AX = I$, with X unknown $n \times n$ matrix.
If $\vec{x}_1, \ldots, \vec{x}_n$ are columns of A, then want to solve n linear systems $A\vec{x}_1 = \vec{e}_1, \ldots, A\vec{x}_n = \vec{e}_n$. Can do so simultaneously using one "super-augmented matrix."