Subspaces, basis, dimension, and rank

Math 40, Introduction to Linear Algebra
Wednesday, February 8, 2012

Subspaces of $\mathbb{R}^n$

One motivation for notion of subspaces of $\mathbb{R}^n$ } algebraic generalization of geometric examples of lines and planes through the origin

$\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$ point (no direction vectors)

$t\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$ line (one direction vector)

$s\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} + t\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$ plane (two direction $\underbrace{\text{vectors)}}_{\text{linearly independent}}$

$r\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix} + s\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} + t\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$ name?

$p\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} + r\begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix} + s\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} + t\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$ name?

Subspace

Definition A *subspace* S of $\mathbb{R}^n$ is a set of vectors in $\mathbb{R}^n$ such that

(1) $\vec{0} \in S$ [contains zero vector]

(2) if $\vec{u}, \vec{v} \in S$, then $\vec{u} + \vec{v} \in S$ [closed under addition]

(3) if $\vec{u} \in S$ and $c \in \mathbb{R}$, then $c\vec{u} \in S$ [closed under scalar mult.]

Subspace

Example

Is $S = \{ \left[\begin{smallmatrix} x \\ y \end{smallmatrix} \right] : x \geq 0, y \geq 0 \}$ a subspace of $\mathbb{R}^2$?

Definition A *subspace* S of $\mathbb{R}^n$ is a set of vectors in $\mathbb{R}^n$ such that

(1) $\vec{0} \in S$

(2) if $\vec{u}, \vec{v} \in S$, then $\vec{u} + \vec{v} \in S$

(3) if $\vec{u} \in S$ and $c \in \mathbb{R}$, then $c\vec{u} \in S$

No!

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in S \text{ but } - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \notin S$$

$\Longrightarrow$ S is not closed under scalar multiplication

Subspace

Example

Is $S = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^3$?

Definition A *subspace* S of $\mathbb{R}^n$ is a set of vectors in $\mathbb{R}^n$ such that

(1) $\vec{0} \in S$

(2) if $\vec{u}, \vec{v} \in S$, then $\vec{u} + \vec{v} \in S$

(3) if $\vec{u} \in S$ and $c \in \mathbb{R}$, then $c\vec{u} \in S$

YES!

(1) $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S \quad \Rightarrow$ contains zero vector ✓

(2) Let $\vec{u}, \vec{v} \in S$. Then $\vec{u} = \begin{bmatrix} a_1 \\ b_1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} a_2 \\ b_2 \\ 0 \end{bmatrix}$ for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$. It follows that

$$\vec{u} + \vec{v} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ 0 \end{bmatrix} \in S \Rightarrow \text{closed under addition} ✓$$

(3) Let $\vec{u} \in S, c \in \mathbb{R}$. Then $\vec{u} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ for some $a, b \in \mathbb{R}$. It follows that

$$c\vec{u} = c \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} ca \\ cb \\ 0 \end{bmatrix} \in S \quad \Rightarrow \text{closed under scalar mult.} ✓$$

Span is a subspace!

Theorem. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{R}^n$. Then $S = \text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$ is a subspace of $\mathbb{R}^n$.

Proof. We verify the three properties of the subspace definition.

(1) $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_k$

$\Rightarrow \vec{0}$ is a linear comb. of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ $\quad \Rightarrow \vec{0} \in S$

(2) Let $\vec{u}, \vec{w} \in S$. Then $\vec{u} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ and $\vec{w} = d_1\vec{v}_1 + \cdots + d_k\vec{v}_k$ for some scalars c_i, d_i. Thus,

$$\begin{aligned} \vec{u} + \vec{w} &= (c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) + (d_1\vec{v}_1 + \cdots + d_k\vec{v}_k) \\ &= \underbrace{(c_1 + d_1)\vec{v}_1 + \cdots + (c_k + d_k)\vec{v}_k}_{\text{linear comb. of } \vec{v}_1, \ldots, \vec{v}_k} \quad \Rightarrow \vec{u} + \vec{w} \in S \end{aligned}$$

(3) Let $\vec{u} \in S, c \in \mathbb{R}$. You finish the proof (show that $c\vec{u} \in S$).

■

Column and row spaces of a matrix

Definition For an $m \times n$ matrix A with column vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^m$, the *column space* of A is $\text{span}(v_1, v_2, \ldots, v_n)$.

col(A) is a subspace of $\mathbb{R}^m$ since it is the span of a set of vectors in $\mathbb{R}^m$

Definition For an $m \times n$ matrix A with row vectors $r_1, r_2, \ldots, r_m \in \mathbb{R}^n$, the *row space* of A is $\text{span}(r_1, r_2, \ldots, r_m)$.

row(A) is a subspace of $\mathbb{R}^n$ since it is the span of a set of vectors in $\mathbb{R}^n$

Characterizing column and row spaces

Important relationships:

- **Column space**

$\vec{b} \in \text{col}(A) \iff$ $\vec{b}$ is in span of cols of A $\iff$ $\exists$ scalars $x_1, x_2, \ldots, x_n$ such that $x_1 \begin{bmatrix} | \\ \vec{v}_1 \\ | \end{bmatrix} + \cdots x_n \begin{bmatrix} | \\ \vec{v}_n \\ | \end{bmatrix} = \vec{b}$ $\iff$ $A\vec{x} = \vec{b}$ is consistent (has a sol'n)

- **Row space**

$\vec{b} \in \text{row}(A) \iff \vec{b}^{\,T} \in \text{col}(A^T) \iff A^T\vec{x} = \vec{b}^{\,T}$ has a solution

since columns of A^T are the rows of A

OR

$\vec{b} \in \text{row}(A) \iff$ $\vec{b}$ is linear comb. of rows of A $\iff \left[\begin{array}{c} A \\ \hline \vec{b} \end{array}\right] \xrightarrow[\text{for } i > j]{R_i + kR_j} \left[\begin{array}{c} A' \\ \hline \vec{0} \end{array}\right]$

Null space of a matrix

Definition For an $m \times n$ matrix A, the *null space* of A is the set of all solutions to $A\vec{x} = \vec{0}$, i.e.,

null(A)

$$\text{null}(A) = \{\vec{x} : A\vec{x} = \vec{0}\}.$$

null(A) is a set of vectors in $\mathbb{R}^n$

Question Is null(A) a subspace of $\mathbb{R}^n$? YES!

This statement requires proof,
and we will tackle this on Friday.

Basis

Definition A set of vectors $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is a *basis* for a subspace S of $\mathbb{R}^n$ if

- span(B) $= S$,
- and B is a linearly independent set.

Example Standard basis for $\mathbb{R}^3$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

but another basis for $\mathbb{R}^3$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

choice of basis is not unique

More on basis

Definition A set of vectors $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is a *basis* for a subspace S of $\mathbb{R}^n$ if

- $\text{span}(B) = S$,
- and B is a linearly independent set.

Standard basis for $\mathbb{R}^3$ is $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$

but another basis for $\mathbb{R}^3$ is $\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$.

Consider

Do you believe such bases exist for $\mathbb{R}^3$?

$$B_1 = \left\{ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}, \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} \right\}$$

$$B_2 = \left\{ \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}, \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}, \begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}, \begin{bmatrix}w_1\\w_2\\w_3\end{bmatrix} \right\}$$

No!

Why not?

- $\text{span}(B_1) \neq \mathbb{R}^3$
- B_2 not linearly indep.

Dimension

proof by contradiction

Theorem. *Any two bases of a subspace have the same number of vectors.*

Definition The number of vectors in a basis of a subspace S is called the *dimension* of S.

Example $\dim(\mathbb{R}^n) = n$

since $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\} = \left\{ \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \ldots, \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}, \right\}$ is a basis for $\mathbb{R}^n$

Side-note The trivial subspace $\{\vec{0}\}$ has no basis since any set containing the zero vector is linearly dependent, so $\dim(\{\vec{0}\}) = 0$.

Important basis results

Theorem. *Given a basis* $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ *of subspace* S, *there is a* **unique** *way to express any* $\vec{v} \in S$ *as a linear combination of basis vectors* $\vec{v}_1, \ldots, \vec{v}_k$.

Proof sketch on Friday.

Theorem. *The vectors* $\{\vec{v}_1, \ldots, \vec{v}_n\}$ *form a basis of* $\mathbb{R}^n$ *if and only if* $\text{rank}(A) = n$, *where* A *is the matrix with columns* $\vec{v}_1, \ldots, \vec{v}_n$.

Proof sketch ($\Rightarrow$).

$\text{rank}(A) = n \Rightarrow$ RREF of A is I

$\nearrow$ $A\vec{x} = \vec{0}$ has only trivial sol'n $\vec{x} = \vec{0}$ $\Rightarrow$ cols of A are linearly indep. $\searrow$

$\searrow$ $A\vec{x} = \vec{b}$ is consistent for any $\vec{b} \in \mathbb{R}^n$ $\Rightarrow$ cols of A span $\mathbb{R}^n$ $\nearrow$

cols of A are basis for $\mathbb{R}^n$

Same ideas can be used to prove converse direction.

Fundamental Theorem of Invertible Matrices (extended)

Theorem. Let A be an $n \times n$ matrix. The following statements are equivalent:

- A is invertible.
- $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^n$.
- $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$.
- The RREF of A is I.
- A is the product of elementary matrices.
- $\text{rank}(A) = n$.
- Columns of A form a basis for $\mathbb{R}^n$.

Finding bases for fundamental subspaces of a matrix

Given matrix A, how do we find bases for subspaces $\begin{cases} \text{row}(A) \\ \text{col}(A) \\ \text{null}(A) \end{cases}$?

First, get RREF of A. $A \xrightarrow{\text{EROs}} R$

Finding bases for fundamental subspaces of a matrix

EROs do not change row space of a matrix.

basis for row(A) = basis for row(R) $\Longrightarrow$ nonzero rows of R

Columns of A have the same dependence relationship as columns of R.

basis for col(A) $\Longrightarrow$ columns of A that correspond to columns of R w/leading 1's

- solve $A\vec{x} = \vec{0}$, i.e. solve $R\vec{x} = \vec{0}$
- express sol'ns in terms of free variables, e.g.,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} \wr \end{bmatrix} + x_3 \begin{bmatrix} \wr \end{bmatrix}$$

basis vectors for null(A)

Example of matrix subspaces' bases

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{bmatrix} \qquad A \xrightarrow{\text{EROs}} R = \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix}\text{basis for} \\ \text{row}(A)\end{matrix} = \left\{ \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix} \right\}$$

$$\begin{matrix}\text{basis for} \\ \text{col}(A)\end{matrix} = \left\{ \begin{bmatrix} 1 \\ 6 \\ 11 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 12 \end{bmatrix} \right\}$$

Example of matrix subspaces' bases

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{bmatrix} \qquad A \xrightarrow{\text{EROs}} R = \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 \qquad - x_3 - 2x_4 - 3x_5 &= 0 \\ x_2 + 2x_3 + 3x_4 + 4x_5 &= 0 \\ x_3, x_4, x_5 \quad &\text{free} \end{aligned}$$

$$\begin{matrix}\text{basis for} \\ \text{null}(A)\end{matrix} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 + 3x_5 \\ -2x_3 - 3x_4 - 4x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Example related to column space

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \qquad \vec{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Is $b \in \text{col}(A)$?
Is $c \in \text{col}(A)$?

Determine if $A\vec{x} = \vec{b}$ has a solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

system is consistent
(has infinite # of sol'ns)

Yes, it is in column space of A.

A solution to the system gives scalar coefficients for linear combination.

$x_1 = 2 - x_3$
$x_2 = 1 + x_3$
x_3 free

one sol'n $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$$\vec{b} = 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Example related to column space

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \qquad \vec{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Is $\vec{b} \in \text{col}(A)$?
Is $\vec{c} \in \text{col}(A)$?

Any vector in the column space of A
has 0 in its third component.

Thus, the vector $\vec{c}$ is not in the column space of A.

Example related to row space

$$A = \begin{bmatrix} -6 & 3 \\ 1 & -\frac{1}{2} \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 2 & 1 \end{bmatrix} \qquad \text{Is } \vec{b} \in \text{row}(A)?$$

Approach 1:

Is $\vec{b}^T \in \text{col}(A^T)$?

Determine if $A^T\vec{x} = \vec{b}^T$ has a solution.

$$\left[\begin{array}{cc|c} -6 & 1 & 2 \\ 3 & -\frac{1}{2} & 1 \end{array}\right] \xrightarrow{R_2 + \frac{1}{2}R_1} \left[\begin{array}{cc|c} -6 & 1 & 2 \\ 0 & 0 & 2 \end{array}\right]$$

inconsistent system

No, $\vec{b} \notin \text{row}(A)$.

Approach 2:

$\vec{b} \in \text{row}(A) \iff$

$$\left[\begin{array}{c} A \\ \hline \vec{b} \end{array}\right] \xrightarrow[\text{for } i>j]{R_i + kR_j} \left[\begin{array}{c} A' \\ \hline \mathbf{0} \end{array}\right]$$

$$\left[\begin{array}{cc} -6 & 3 \\ 1 & -\frac{1}{2} \\ \hline 2 & 1 \end{array}\right] \xrightarrow[R_2 + \frac{1}{6}R_1]{R_3 - 2R_2} \left[\begin{array}{cc} -6 & 3 \\ 0 & 0 \\ \hline 0 & 2 \end{array}\right]$$

No, $\vec{b} \notin \text{row}(A)$.

Example related to null space

$$A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & -2 & 1 & 1 \\ 4 & -4 & 3 & -1 \end{bmatrix}$$

Find null(A).

We need to solve $A\vec{x} = \vec{0}$.

We have

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 0 \\ 2 & -2 & 1 & 1 & 0 \\ 4 & -4 & 3 & -1 & 0 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_2, x_4 free vars

Convert to equations.

$$x_1 - x_2 + 2x_4 = 0$$
$$x_3 - 3x_4 = 0$$

Solve for x_1 and x_3.

$$\vec{x} = \begin{bmatrix} x_2 - 2x_4 \\ x_2 \\ 3x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

for $x_2, x_4 \in \mathbb{R}$

$$\text{null}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right)$$