Rank and linear transformations

Math 40, Introduction to Linear Algebra
Friday, February 10, 2012

Recall from Wednesday....

Important characteristic of a basis

Theorem. *Given a basis* $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ *of subspace* S*, there is a* **unique** *way to express any* $\vec{v} \in S$ *as a linear combination of basis vectors* $\vec{v}_1, \ldots, \vec{v}_k$*.*

Proof sketch. Suppose

span$(B) = S$

$$\begin{aligned} \vec{v} &= c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \\ \vec{v} &= d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_k\vec{v}_k \end{aligned} \quad \text{for scalars } c_i, d_i$$

$$\text{Then } \vec{0} = (c_1 - d_1)\vec{v}_1 + (c_2 - d_2)\vec{v}_2 + \cdots + (c_k - d_k)\vec{v}_k$$

linearly independent

$$\Rightarrow \left.\begin{array}{c} c_1 - d_1 = 0 \\ c_2 - d_2 = 0 \\ \vdots \\ c_k - d_k = 0 \end{array}\right\rangle c_i = d_i \;\; \forall i$$

$\blacksquare$

Recall from Wednesday....

Example of matrix subspaces' bases

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{bmatrix} \qquad A \xrightarrow{\text{EROs}} R = \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix}\text{basis for} \\ \text{row}(A)\end{matrix} = \left\{ \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix} \right\}$$

$$\begin{matrix}\text{basis for} \\ \text{col}(A)\end{matrix} = \left\{ \begin{bmatrix} 1 \\ 6 \\ 11 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 12 \end{bmatrix} \right\}$$

Example of matrix subspaces' bases

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{bmatrix} \qquad A \xrightarrow{\text{EROs}} R = \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 \qquad - \quad x_3 - 2x_4 - 3x_5 &= 0 \\ x_2 + 2x_3 + 3x_4 + 4x_5 &= 0 \\ x_3, x_4, x_5 \quad &\text{free} \end{aligned}$$

$$\boxed{\begin{matrix}\text{basis for} \\ \text{null}(A)\end{matrix} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 + 3x_5 \\ -2x_3 - 3x_4 - 4x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Dimensions of fundamental subspaces

For the 3 x 5 matrix from the last example, we have

$$\text{basis for row}(A) = \left\{ \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix} \right\}$$

$$\text{basis for col}(A) = \left\{ \begin{bmatrix} 1 \\ 6 \\ 11 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 12 \end{bmatrix} \right\} \qquad \text{basis for null}(A) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Note that $\dim(\text{row}(A)) = 2 \longleftarrow$ ***row rank*** of A

$\dim(\text{col}(A)) = 2 \longleftarrow$ ***column rank*** of A

$\dim(\text{null}(A)) = 3 \longleftarrow$ ***nullity*** of A

Question: Is it a coincidence that row rank of A = column rank of A ?

Question: Is it a coincidence that nullity of A + row/col rank of A = # of cols of A?

Rank of a matrix

Theorem. *For any matrix A,*

$$\underbrace{\textit{row rank of } A = \textit{column rank of } A.}_{\textit{rank of } A}$$

# of nonzero rows of RREF of A

# of leading 1's in RREF of A

Consequence: $\text{rank}(A) = \text{rank}(A^T)$

Theorem (Rank Theorem)**.** *For any $m \times n$ matrix A,*

$$\text{nullity}(A) + \text{rank}(A) = n.$$

# of columns without leading 1's in RREF of A

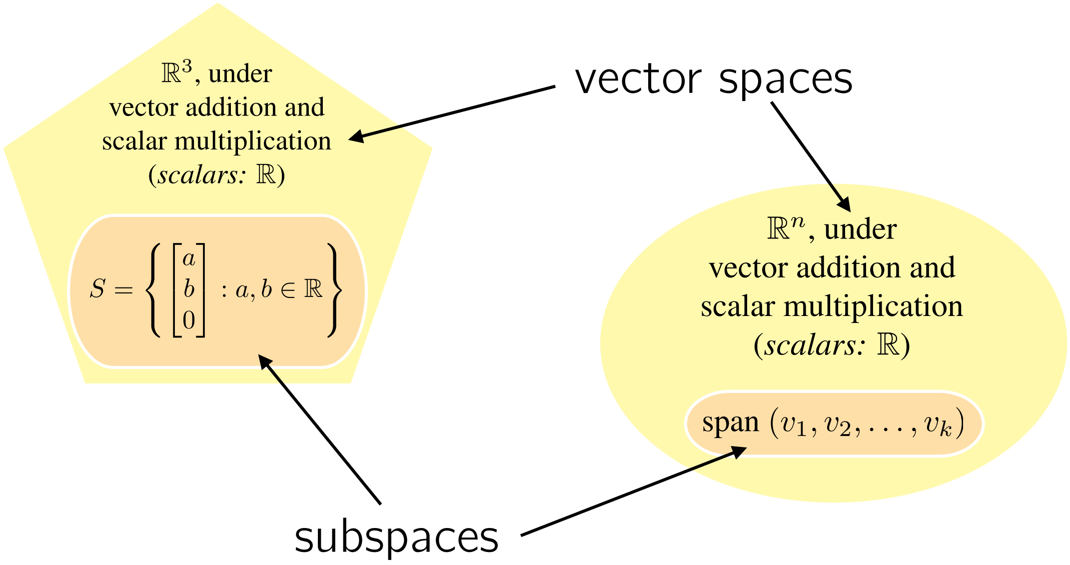
# of columns with leading 1's in RREF of A

Fundamental Theorem of Invertible Matrices (extended)

Theorem. Let A be an n x n matrix. The following statements are equivalent:

- A is invertible.
- $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^n$.
- $A\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$.
- The RREF of A is I.
- A is the product of elementary matrices.
- rank(A) = n.
- nullity(A) = 0.
- Columns of A are linearly independent.
- Columns of A span $\mathbb{R}^n$.
- Columns of A form a basis for $\mathbb{R}^n$.
- Rows of A are linearly independent.
- Rows of A span $\mathbb{R}^n$.
- Rows of A form a basis for $\mathbb{R}^n$.

Examples of vector spaces and subspaces



$\mathbb{R}^3$, under vector addition and scalar multiplication (*scalars:* $\mathbb{R}$)

$S = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$

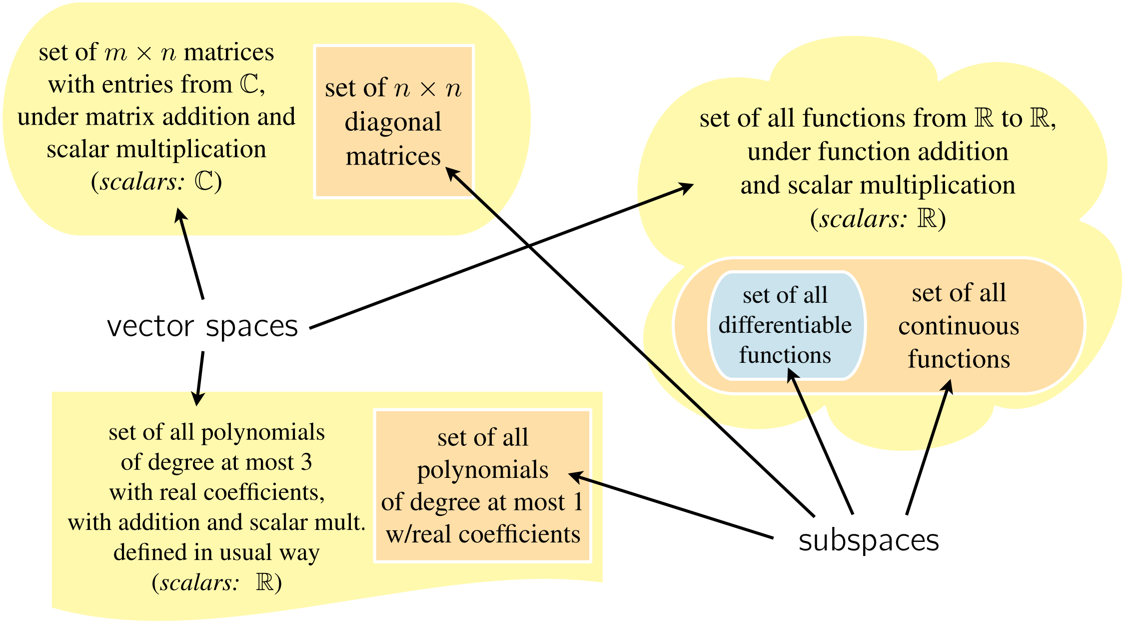
vector spaces

$\mathbb{R}^n$, under vector addition and scalar multiplication (*scalars:* $\mathbb{R}$)

span $(v_1, v_2, \ldots, v_k)$

subspaces

Thinking beyond Euclidean vectors:
more examples of vector spaces and subspaces



set of $m \times n$ matrices with entries from $\mathbb{C}$, under matrix addition and scalar multiplication (*scalars:* $\mathbb{C}$)

set of $n \times n$ diagonal matrices

set of all functions from $\mathbb{R}$ to $\mathbb{R}$, under function addition and scalar multiplication (*scalars:* $\mathbb{R}$)

set of all differentiable functions

set of all continuous functions

vector spaces

set of all polynomials of degree at most 3 with real coefficients, with addition and scalar mult. defined in usual way (*scalars:* $\mathbb{R}$)

set of all polynomials of degree at most 1 w/real coefficients

subspaces

A central idea of linear algebra: linear transformations

Definition A function $T:\mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* if $\forall \vec{u}, \vec{v} \in \mathbb{R}^n$ and scalars c,

- $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$,
- and $T(c\vec{v}) = cT(\vec{v})$.

streamlined as $T(c_1\vec{u}+c_2\vec{v}) = c_1T(\vec{u}) + c_2T(\vec{v})$

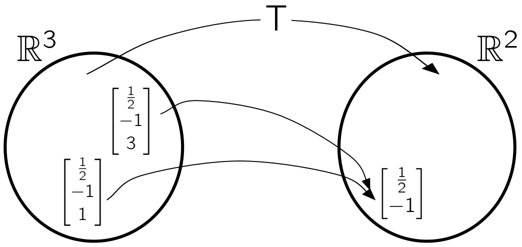
Example

$$T:\underset{\text{domain}}{\mathbb{R}^3} \to \underset{\text{codomain}}{\mathbb{R}^2} \quad \text{and} \quad T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} \frac{1}{2} \\ -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} \frac{1}{2} \\ -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix}$$



A central idea of linear algebra: linear transformations

Definition A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* if $\forall \vec{u}, \vec{v} \in \mathbb{R}^n$ and scalars c,

$$\left.\begin{array}{l} \bullet\ T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}),\\ \bullet\ \text{and } T(c\vec{v}) = cT(\vec{v}).\end{array}\right\} \quad \begin{array}{c}\text{streamlined as}\\ T(c_1\vec{u}+c_2\vec{v}) = c_1T(\vec{u}) + c_2T(\vec{v})\end{array}$$

Non-example

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \text{and} \qquad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ xy \end{bmatrix}$$

$$\text{Consider} \quad T\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \neq 2\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = 2\,T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

Kernel and range of linear transformation

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

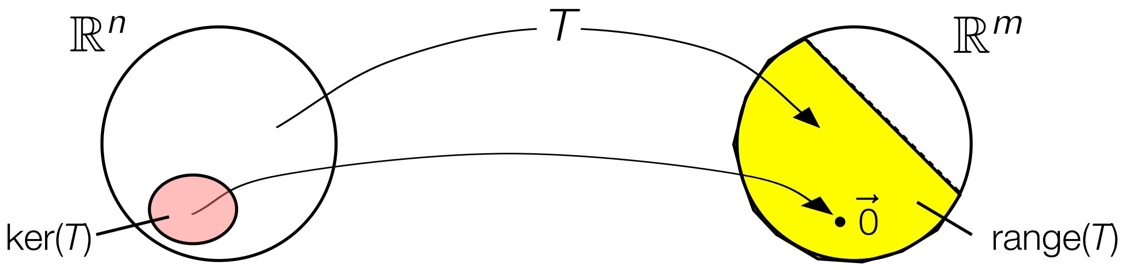
Definition The *kernel* or *null space* of T is

$$\ker(T) = \{\vec{v} \in \mathbb{R}^n : T(\vec{v}) = \vec{0}\}.$$

The *image* or *range* of T is

set of all images under T

$$\text{range}(T) = \{\vec{w} \in \mathbb{R}^m : \vec{w} = T(\vec{v}) \text{ for some } \vec{v} \in \mathbb{R}^n\}.$$



Looking closer at an example

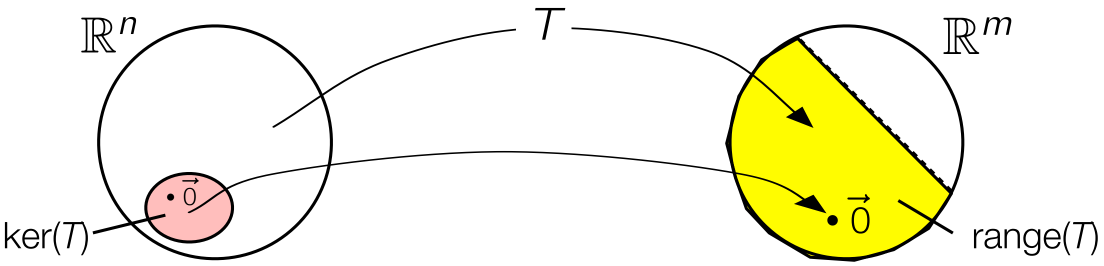
Example $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\ker(T) = \left\{\vec{v} \in \mathbb{R}^3 : T(\vec{v}) = \vec{0}\right\} = \left\{\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} : z \in \mathbb{R}\right\} = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

For any $x, y \in \mathbb{R}$, $T\left(\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$ $\Longrightarrow$ so $\text{range}(T) = \mathbb{R}^2$

subspaces!

Remarks on linear transformations



For any linear transformation T,

- $T(\vec{0}) = \vec{0}$,
- ker(T) is a subspace of $\mathbb{R}^n$,
- range(T) is a subspace of $\mathbb{R}^m$.

We define

nullity(T) = dim(ker(T))

rank(T) = dim(range(T))

Theorem (Rank Thm for linear transformations)**.** *For a linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$, $\text{rank}(T) + \text{nullity}(T) = \dim(\mathbb{R}^n) = n$.

Matrix multiplication as a linear transformation

Primary example of a linear transformation $\Longrightarrow$ matrix multiplication

Given an $m \times n$ matrix A,
define $T(\vec{x}) = A\vec{x}$ for $\vec{x} \in \mathbb{R}^n$.

Then T is a linear transformation.

Astounding!

Matrix multiplication defines a linear transformation.

This new perspective gives a dynamic view of a matrix (it transforms vectors into other vectors) and is a key to building math models to physical systems that evolve over time (so-called dynamical systems).

Matrix multiplication as a mapping (or function)

Given an $m \times n$ matrix A,
define $T(\vec{x}) = A\vec{x}$ for $\vec{x} \in \mathbb{R}^n$.

Verify that T is a linear transformation.

We have

$$T(\vec{u}+\vec{v}) \;= A(\vec{u}+\vec{v}) \;= A\vec{u}+A\vec{v} \;= T(\vec{u})+T(\vec{v}) \quad ✓$$

and

$$T(c\vec{v}) \;= A(c\vec{v}) \;= c(A\vec{v}) \;= cT(\vec{v}) \quad ✓$$

What is range of T?

$$\text{range}(T) = \begin{array}{c}\text{set of all vectors } \vec{b} \in \mathbb{R}^m \\ \text{such that } T(\vec{x}) = \vec{b} \text{ for} \\ \text{some } \vec{x} \in \mathbb{R}^n\end{array} = \begin{array}{c}\text{set of all vectors } \vec{b} \in \mathbb{R}^m \\ \text{such that } A\vec{x} = \vec{b} \text{ for} \\ \text{some } \vec{x} \in \mathbb{R}^n\end{array} = \text{col}(A)$$

Similarly, $\ker(T) = \text{null}(A)$

A linear transformation as matrix multiplication

Theorem. *Every linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ *can be represented by an* $m \times n$ *matrix* A *so that* $\forall\, \vec{x} \in \mathbb{R}^n$,

$$T(\vec{x}) = A\vec{x}.$$

More astounding!

Question Given T, how do we find A?

Transformation T is completely determined by its action on basis vectors.

Consider standard basis vectors for $\mathbb{R}^n$:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ldots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Compute $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$.

Standard matrix of a linear transformation

Question Given T, how do we find A?

Transformation T is completely determined by its action on basis vectors.

Consider standard basis vectors for $\mathbb{R}^n$:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ldots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Compute $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$.

Then
$$A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix}$$
is called the *standard matrix* for T.

Standard matrix for an example

Example $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

What is $T\left(\begin{bmatrix} 2 \\ -5 \\ 12 \end{bmatrix}\right)$?

$$\Rightarrow A\begin{bmatrix} 2 \\ -5 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \\ 12 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -5 \end{bmatrix}}$$