Linear transformations and determinants

Math 40, Introduction to Linear Algebra
Monday, February 13, 2012

Matrix multiplication as a linear transformation

Primary example of a linear transformation $\Longrightarrow$ matrix multiplication

Given an $m \times n$ matrix A,
define $T(\vec{x}) = A\vec{x}$ for $\vec{x} \in \mathbb{R}^n$.

Then T is a linear transformation.

Astounding!

Matrix multiplication defines a linear transformation.

This new perspective gives a dynamic view of a matrix (it transforms vectors into other vectors) and is a key to building math models to physical systems that evolve over time (so-called dynamical systems).

A linear transformation as matrix multiplication

Theorem. *Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be represented by an $m \times n$ matrix A so that $\forall\, \vec{x} \in \mathbb{R}^n$,*

$$T(\vec{x}) = A\vec{x}.$$

Question Given T, how do we find A?

Transformation T is completely determined by its action on basis vectors.

Consider standard basis vectors for $\mathbb{R}^n$:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ldots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Compute $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$.

Standard matrix of a linear transformation

Question Given T, how do we find A?

Transformation T is completely determined by its action on basis vectors.

Consider standard basis vectors for $\mathbb{R}^n$:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ldots, \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Compute $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$.

Then

$$\begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix}$$

is called the *standard matrix* for T.

denoted $[\,T\,]$

Standard matrix for an example

Example $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

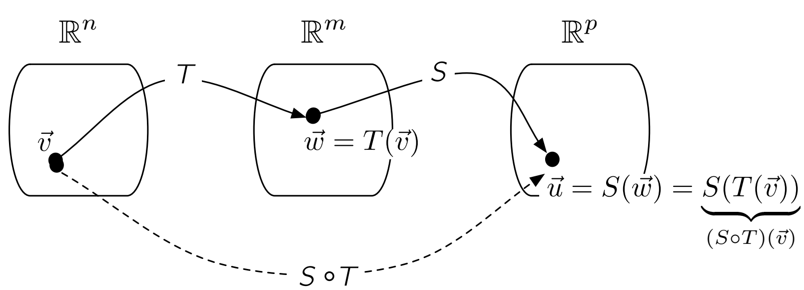
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

What is $T\left(\begin{bmatrix} 2 \\ -5 \\ 12 \end{bmatrix}\right)$?

$$\Rightarrow A\begin{bmatrix} 2 \\ -5 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

Composition

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with standard matrix A
$S : \mathbb{R}^m \to \mathbb{R}^p$ is a linear transformation with standard matrix B.



The *composition* $S \circ T : \mathbb{R}^n \to \mathbb{R}^p$ is a linear transformation and

since A is standard matrix of transformation T

$$(S \circ T)(\vec{x}) = S(T(\vec{x})) = S(A\vec{x}) = B(A\vec{x}) = BA\vec{x} = [S][T]\,\vec{x} \qquad \Longrightarrow \; [S \circ T] = [S][T]$$

since B is standard matrix of transformation S

Inverse

We say S and T are *inverse linear transformations*
if $S \circ T = I$ and $T \circ S = I$,
where I is the identity transformation.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation. Then its standard matrix $[T]$ is an invertible matrix, and

$$[T^{-1}] = [T]^{-1}.$$

The matrix of the inverse is the inverse of the matrix!

Introduction to determinants: 2x2 case

The determinant is only defined for a **square** matrix.

2 x 2 matrices

Definition For 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the *determinant* of A is

$$\det(A) = ad - bc.$$

also denoted $|A|$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Key property:

$$A \text{ invertible} \iff \det(A) = ad - bc \neq 0$$

Introduction to determinants: 3x3 case

3 x 3 matrices

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then we have

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \xrightarrow{\text{EROs}} \begin{bmatrix} a & b & c \\ 0 & ae-bd & af-cd \\ 0 & 0 & a\Delta \end{bmatrix}$$

assume $a \neq 0$
and $ae - bd \neq 0$

A invertible
$\Updownarrow$
$\Delta \neq 0$

where $\Delta = (aei - ahf) - (bdi - bgf) + (cdh - cge)$

Observe that $\Delta = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

How do these relate to A?

$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ A_{11}, submatrix formed by deleting row 1, col 1

$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ A_{12}, submatrix formed by deleting row 1, col 2

$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ A_{13}

Definition of determinant

3 x 3 matrices

$$\Delta = \underbrace{a}_{a_{11}} \det A_{11} - \underbrace{b}_{a_{12}} \det A_{12} + \underbrace{c}_{a_{13}} \det A_{13}$$

n x n matrices

For an n x n matrix, the ***determinant*** of A is

$$\det(A) = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{n+1} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{j+1} \; a_{1j} \; \det A_{1j}$$

signs of terms alternate

entries of 1st row of A

determinants of $(n\text{-}1)$ x $(n\text{-}1)$ submatrices formed by deleting 1st row and jth col

So the determinant is defined recursively.

Example of computing the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= (5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7)$$

$$= -3 + 12 - 9$$

$$= \boxed{0}$$

$\implies$ matrix is **not** invertible

Amazing facts about determinants

★ det A can be found by "expanding" along **any** row or any column

original def'n expands across row 1

use row i $\implies$ $\det A = \sum_{j=1}^{n} (-1)^{i+j} \, a_{ij} \, \det A_{ij}$ [i is fixed, j varies]

use col j $\implies$ $\det A = \sum_{i=1}^{n} (-1)^{i+j} \, a_{ij} \, \det A_{ij}$ [j is fixed, i varies]

Amazing facts about determinants

★ det A can be found by "expanding" along **any** row or any column

sign pattern

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & & \\ + & - & + & & & \\ - & + & & \ddots & & \\ + & & & & & \\ \vdots & & & & & \end{bmatrix}$$

negative when indices sum to odd

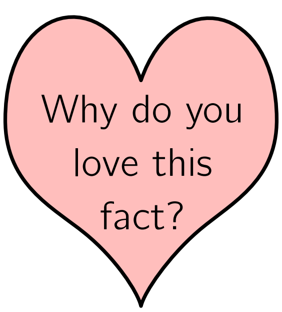
$$\underbrace{(-1)^{i+j}\det A_{ij}}$$

(i, j)-cofactor of A

$$\left[\det A = \sum a_{ij} C_{ij}\right]$$

Amazing facts about determinants

★ det A can be found by "expanding" along **any** row or any column



Example

$$A = \begin{bmatrix} 2 & 1 & 7 \\ -1 & 3 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det A = 0 \begin{vmatrix} * & * \\ * & * \end{vmatrix} - 1 \begin{vmatrix} 2 & 7 \\ -1 & 2 \end{vmatrix} + 0 \begin{vmatrix} * & * \\ * & * \end{vmatrix}$$

$$= -(2 \cdot 2 - 7(-1))$$

$$= \boxed{-11}$$

Amazing facts about determinants

⭐ det A can be found by "expanding" along **any** row or any column

Consequence: **Theorem.** *The determinant of a triangular matrix is the product of its diagonal entries.*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$\det(A) = 1 \cdot 5 \cdot 8 \cdot 10 = \boxed{400}$$

Amazing facts about determinants

⭐ EROs barely change the determinant, and they do so in a predictable way.

EROs	effect on det A
swap two rows	changes sign
multiply row by scalar c	multiply det by scalar c
add $c \cdot$ row i to row j	no change at all!

Strategy to compute det A more quickly for general matrices A

$\Downarrow$

Perform EROs to get REF of A and compute det A based on det of REF