Determinants and eigenvalues

Math 40, Introduction to Linear Algebra
Wednesday, February 15, 2012

Amazing facts about determinants

★ det A can be found by "expanding" along **any** row or any column

Consequence: **Theorem.** *The determinant of a triangular matrix is the product of its diagonal entries.*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

$$\det(A) = 1 \cdot 5 \cdot 8 \cdot 10 = \boxed{400}$$

Amazing facts about determinants

★ EROs barely change the determinant, and they do so in a predictable way.

EROs	effect on det A
swap two rows	changes sign
multiply row by scalar c	multiply det by scalar c
add $c \cdot$ row i to row j	no change at all!

Strategy to compute det A more quickly for general matrices A

$\Downarrow$

Perform EROs to get REF of A and compute det A based on det of REF

Amazing facts about determinants

★ **Theorem.** *A square matrix A is invertible if and only if* $\det A \neq 0$.

★ $\det(A) = \det(A^T)$

★ $\det(AB) = \det(A)\det(B)$ (determinant is multiplicative)

Consequence:

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det A}$$

★ If A is invertible, $\det(A^{-1}) = \frac{1}{\det A}$.

Example using properties of determinant

Example If det A = -3 for a 5 x 5 matrix A, find the determinant of the matrix $4A^3$.

We have

$$\det(4A^3) = 4^5 \det(A^3) = 4^5[\det(A)]^3 = \boxed{4^5(-3)^3}$$

Another property of the determinant?

Question True or false:

$$\det(A + B) = \det A + \det B \; ?$$

False!

Consider

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \det(A + B) = 4 \neq 0 = \det A + \det B$$

Eigenvalues and eigenvectors

Introduction to eigenvalues

Let A be an n x n matrix.

If $A\vec{x} = \lambda\vec{x}$ for some scalar λ
and some nonzero vector $\vec{x}$,

then we say λ is an *eigenvalue* of A
and $\vec{x}$ is an *eigenvector* associated with λ.

Viewed as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$
A sends vector $\vec{x}$ to a scalar multiple of itself $(\lambda\vec{x})$.

Eigenvalues, eigenvectors for a 2x2 matrix

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 12 \\ 30 \end{bmatrix} = 6 \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

eigenvectors   eigenvalues

$$\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Notice that

$$\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 15 \end{bmatrix} = 6 \begin{bmatrix} 6 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ 15 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

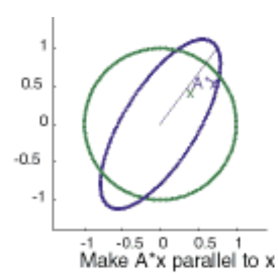
Any (nonzero) scalar multiple of an eigenvector is itself an eigenvector (associated w/same eigenvalue).

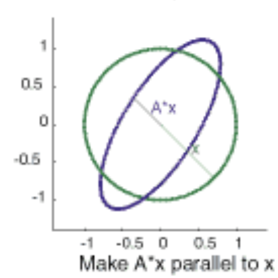
$$A(c\vec{x}) = c(A\vec{x}) = c(\lambda\vec{x}) = \lambda(c\vec{x})$$

Graphic demonstration of eigenvalues and eigenvectors: eigshow

eigshow demonstrates how the image $A\vec{x}$ changes as we rotate a unit vector $\vec{x}$ in $\mathbb{R}^2$ around a circle

in particular, we are interested in knowing when $A\vec{x}$ is parallel to $\vec{x}$





Finding eigenvalues of A

We want **nontrivial** solutions to

$$A\vec{x} = \lambda\vec{x} \iff A\vec{x} - \lambda\vec{x} = \vec{0} \iff A\vec{x} - \lambda I\vec{x} = \vec{0}$$

$$\iff (A - \lambda I)\vec{x} = \vec{0}$$

When does this homogeneous system have a solution other than $\vec{x} = \vec{0}$?

Must have that $A - \lambda I$ is not invertible,
which means that $\det(A - \lambda I) = 0$

eigenvalues of A
$\Downarrow$
find values of λ
such that $\det(A - \lambda I) = 0$

given eigenvalue λ,
associated eigenvectors are
nonzero vectors in
$\text{null}(A - \lambda I)$

Example of finding eigenvalues and eigenvectors

Example Find eigenvalues and corresponding eigenvectors of A.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 2 & -1-\lambda & 5 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(1-\lambda)(-1-\lambda)$$

characteristic polynomial
$-\lambda^3 + 2\lambda^2 + \lambda - 2$

$$\boxed{\lambda = 2, 1, \text{or } -1}$$

Example of finding eigenvalues and eigenvectors

Example Find eigenvalues and corresponding eigenvectors of A.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$\lambda = 2, 1, \text{or } -1$

$\lambda = 2$ Solve $(A - 2I)\vec{x} = \vec{0}$.

$$[\, A - 2I \mid 0 \,] = \left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 2 & -3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

for any $x_3 \in \mathbb{R}$

eigenvectors of A for $\lambda = 2$ are

$$c \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \text{ for } c \neq 0$$

Example of finding eigenvalues and eigenvectors

Example Find eigenvalues and corresponding eigenvectors of A.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$\lambda = 2, 1, \text{or } -1$

$\lambda = 2$ Solve $(A - 2I)\vec{x} = \vec{0}$.

eigenvectors of A for $\lambda = 2$ are $c \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ for $c \neq 0$

$$E_2 = \begin{matrix} \textit{eigenspace} \text{ of } A \\ \text{for } \lambda = 2 \end{matrix} = \left\{ \begin{matrix} \text{set of all eigenvectors} \\ \text{of } A \text{ for } \lambda = 2 \end{matrix} \right\} \cup \{\vec{0}\}$$

$$= \text{null}(A - 2I)$$

$$= \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right)$$

Example of finding eigenvalues and eigenvectors

Example Find eigenvalues and corresponding eigenvectors of A.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$\lambda = 2, 1, \text{or } -1$

$\lambda = 2$ $\quad E_2 = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right)$

$\lambda = 1$ $\quad$ Solve $(A - I)\vec{x} = \vec{0}$. $\implies$ $E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$

$\lambda = -1$ $\quad$ Solve $(A + I)\vec{x} = \vec{0}$. $\implies$ $E_{-1} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

Eigenvalues, eigenvalues... where are you?

Example Find eigenvalues of A.

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1$$

$$= \lambda^2 - 2\lambda + 2$$

$$\implies \lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Eigenvalues are complex!