Eigenvalues and eigenvectors

Math 40, Introduction to Linear Algebra
Friday, February 17, 2012

Introduction to eigenvalues

Let A be an n x n matrix.

If $A\vec{x} = \lambda\vec{x}$ for some scalar λ
and some nonzero vector $\vec{x}$,

then we say λ is an *eigenvalue* of A
and $\vec{x}$ is an *eigenvector* associated with λ.

Viewed as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$
A sends vector $\vec{x}$ to a scalar multiple of itself $(\lambda\vec{x})$.

Example of finding eigenvalues and eigenvectors

Example Find eigenvalues and corresponding eigenvectors of A.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 2 & -1-\lambda & 5 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix}$$

characteristic polynomial $-\lambda^3 + 2\lambda^2 + \lambda - 2$

$$= (2-\lambda)(1-\lambda)(-1-\lambda)$$

$$\boxed{\lambda = 2, 1, \text{or } -1}$$

Example of finding eigenvalues and eigenvectors

Example Find eigenvalues and corresponding eigenvectors of A.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\boxed{\lambda = 2, 1, \text{or } -1}$$

$\lambda = 2$ Solve $(A - 2I)\vec{x} = \vec{0}$.

$$[\, A - 2I \mid 0 \,] = \left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 2 & -3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

for any $x_3 \in \mathbb{R}$

eigenvectors of A for $\lambda = 2$ are

$$c \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \text{for } c \neq 0$$

Example of finding eigenvalues and eigenvectors

Example Find eigenvalues and corresponding eigenvectors of A.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$\lambda = 2, 1, \text{or } -1$

$\lambda = 2$ Solve $(A - 2I)\vec{x} = \vec{0}$. eigenvectors of A for $\lambda = 2$ are $c \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ for $c \neq 0$

$$E_2 = \underset{\text{for } \lambda = 2}{\text{eigenspace of } A} = \left\{ \begin{matrix} \text{set of all eigenvectors} \\ \text{of } A \text{ for } \lambda = 2 \end{matrix} \right\} \cup \{\vec{0}\}$$

$$= \text{null}(A - 2I)$$

$$= \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right)$$

Example of finding eigenvalues and eigenvectors

Example Find eigenvalues and corresponding eigenvectors of A.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$\lambda = 2, 1, \text{or } -1$

$\lambda = 2$ $E_2 = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right)$

$\lambda = 1$ Solve $(A - I)\vec{x} = \vec{0}$. $\Longrightarrow$ $E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)$

$\lambda = -1$ Solve $(A + I)\vec{x} = \vec{0}$. $\Longrightarrow$ $E_{-1} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$

Eigenvalues, eigenvalues... where are you?

Example Find eigenvalues of A.

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1$$

$$= \lambda^2 - 2\lambda + 2$$

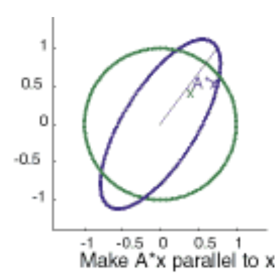
$$\Longrightarrow \lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

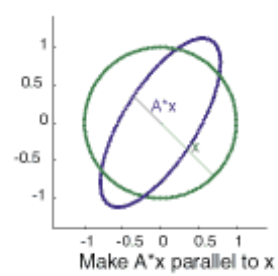
Eigenvalues are complex!

Graphic demonstration of eigenvalues and eigenvectors: eigshow

eigshow demonstrates how the image $A\vec{x}$ changes as we rotate a unit vector $\vec{x}$ in $\mathbb{R}^2$ around a circle

in particular, we are interested in knowing when $A\vec{x}$ is parallel to $\vec{x}$





Algebraic and geometric multiplicities

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix} \qquad 0 = \det(A - \lambda I) = -(\lambda - 3)^3 \implies \lambda = 3$$

Since $\lambda = 3$ appears 3 times as a root of characteristic polynomial, the ***algebraic multiplicity*** of $\lambda = 3$ is 3.

To find eigenspace E_3, we solve $(A - 3I)\vec{x} = \vec{0}$.

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{bmatrix} \xrightarrow{\text{EROs}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_3 = \left\{ \begin{bmatrix} -x_3 \\ x_2 \\ x_3 \end{bmatrix} : x_2, x_3 \in \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Since eigenspace E_3 has dimension 2, the ***geometric multiplicity*** of $\lambda = 3$ is 2.

$\dim(\text{null}(A - \lambda I))$ → geometric multiplicity

Algebraic and geometric multiplicities

Fact:

algebraic multiplicity of an eigenvalue $\geq$ geometric multiplicity of an eigenvalue

Incredible facts about eigenvalues

★ An $n \times n$ matrix has n eigenvalues, including the multiplicities of repeated eigenvalues.

Fundamental Thm of Algebra

★ Eigenvalues of a triangular matrix are the diagonal entries.

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{22} - \lambda & & \cdots & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & a_{nn} - \lambda \end{bmatrix}$$

$$0 = \det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda)$$

$$\Longrightarrow \lambda = a_{11}, a_{22}, \ldots, a_{nn}$$

Incredible facts about eigenvalues

★ $\det A = \underbrace{\lambda_1 \lambda_2 \cdots \lambda_n}_{\text{product of all eigenvalues of } A}$

$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$
evaluate polynomial at $\lambda = 0$

A is invertible $\Longleftrightarrow \det A \neq 0 \Longleftrightarrow$ 0 is not an eigenvalue of A

★ Eigenvalues of A^k and A^{-1}, when it exists, are directly related to eigenvalues of A.

λ is an eigenvalue of A $\Longrightarrow$ λ^k is an eigenvalue of A^k

A invertible, λ is an eigenvalue of A $\Longrightarrow$ $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}

eigenvectors are the same as those associated with λ for A

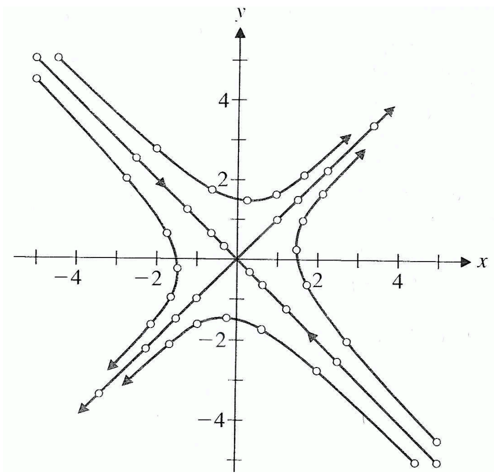
An application

$$A = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

Given $x \in \mathbb{R}^2$, what happens to $A^k x$ as $k \to \infty$?

eigenvalues are 1.5 and 0.5 w/eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, respectively

form a basis



$$\Rightarrow x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ for some } c_1, c_2 \in \mathbb{R}$$

$$\begin{aligned} A^k x &= A^k \left(c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) \\ &= c_1 A^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 A^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \underbrace{c_1 (1.5)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{approaches line } y = x \text{ if } c_1 \neq 0} + \underbrace{c_2 (0.5)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{goes to 0 as } k \to \infty} \end{aligned}$$

Incredible facts about eigenvalues

- ⭐ Distinct eigenvalues of A have linearly independent eigenvectors.
- ⭐ Distinct eigenvalues of a symmetric matrix A have eigenvectors that are orthogonal to each other.