Similarity and diagonalizability

Assume throughout today's discussion that all matrices are square ($n \times n$).

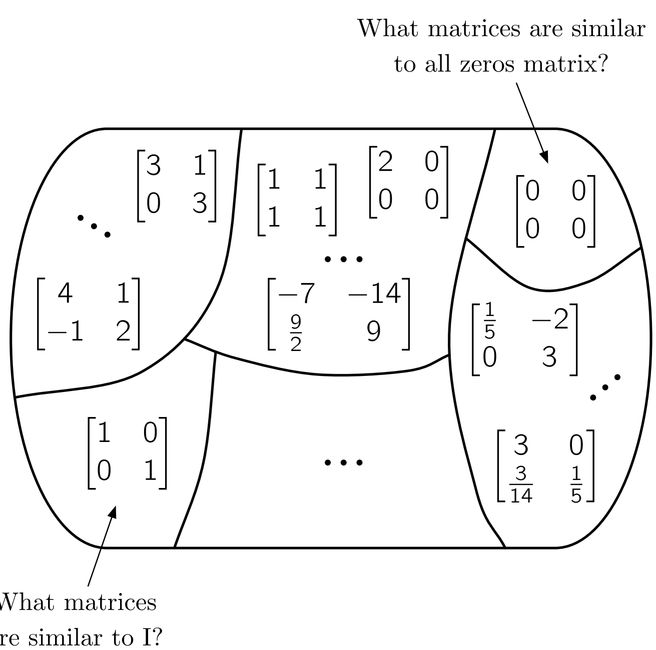
Definition A matrix A is *similar* to B if there exists an invertible matrix P such that $P^{-1}AP = B$. (We denote as $A \sim B$.)

Remarks Similar matrices have the same

- determinant,
  $\det B = \det(P^{-1}AP) = \det(P^{-1}) \ \det A \ \det P = \det A \ \frac{1}{\det P} \ \det P = \det A$
- rank,
- characteristic polynomial,
- and eigenvalues.

Similarity is an equivalence relation that partitions the set of all $n \times n$ matrices into classes.

Example 2×2 matrices



Down the road, you will learn that a representative matrix of each class is called the Jordan canonical form.

Ideally, given a matrix, we would like it to be similar to a....diagonal matrix! (Then eigenvalues would be precisely the diagonal entries of the diagonal matrix to which it is similar.)

Definition A matrix A is *diagonalizable* if it is similar to a diagonal matrix D, i.e.,

$$P^{-1}AP = D \qquad \text{for some invertible matrix } P \text{ and some diagonal matrix } D$$

$$\implies \quad AP = PD.$$

Example The matrix $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ is diagonalizable since $A \sim D$ where $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Check that

$$\underbrace{\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}}_{D},$$

and P is invertible because $\det P \neq 0$.

Example We claim that the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable. Suppose, by way of contradiction (BWOC), that A is diagonalizable, i.e., there exist invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$. Then $A = PDP^{-1}$, and we have

$$\begin{aligned} 0 = A^2 = (PDP^{-1})(PDP^{-1}) &= PD^2P^{-1} \\ &\implies 0 = D^2 \\ &\implies 0 = D \\ &\implies A = PDP^{-1} = P\,0\,P^{-1} = 0 \quad \Rightarrow\Leftarrow \end{aligned}$$

We have arrived at a contradiction, so it must be that A is not diagonalizable.

Questions

- When is A diagonalizable?
- If A is diagonalizable, how do we find matrices P and D such that $P^{-1}AP = D$?

Given matrix A, suppose A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with corresponding eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$. Then

$$\left.\begin{aligned} A\vec{x}_1 &= \lambda_1\vec{x}_1 \\ A\vec{x}_2 &= \lambda_2\vec{x}_2 \\ &\vdots \\ A\vec{x}_n &= \lambda_n\vec{x}_n \end{aligned}\right\}$$

$$\begin{aligned} A \begin{bmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{bmatrix} &= \begin{bmatrix} | & | & & | \\ \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 & \cdots & \lambda_n\vec{x}_n \\ | & | & & | \end{bmatrix} \\ &= \begin{bmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \\ \implies \quad AP &= PD \end{aligned}$$

When is P invertible? $\implies$ eigenvectors of A are linearly independent

Theorem. *Let A be an $n \times n$ matrix. Then*

A is diagonalizable $\iff$ *A has n linearly independent eigenvectors.*