Diagonalizability
Encodings, coordinates, and change of basis

Math 40, Introduction to Linear Algebra
Wednesday, February 22, 2012

Characterization of diagonalizability

Definition A matrix A is *diagonalizable* if it is similar to a diagonal matrix D, i.e.,

$$P^{-1}AP = D$$

for some invertible matrix P and some diagonal matrix D

Theorem. *Let A be an $n \times n$ matrix. Then*

A is diagonalizable $\iff$ *A has n linearly independent eigenvectors.*

If we know that A is diagonalizable, then $P^{-1}AP = D$, and

- columns of P are n linearly independent eigenvectors of A,
- and diagonal entries of D are eigenvalues of A.

Example of diagonalization

Example Consider $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{bmatrix}$.

- Find eigenvalues of A.

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= (3-\lambda)[(4-\lambda)(5-\lambda) - 2] \\ &= (3-\lambda)(\lambda^2 - 9\lambda + 18) \\ &= (3-\lambda)(\lambda-3)(\lambda-6) \end{aligned}$$

$\lambda = 3$ (alg. mult. 2),
$\lambda = 6$

- Find eigenspaces of A.

Solve $(A - 3I)\vec{x} = \vec{0}$. $\implies$ $E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$

Solve $(A - 6I)\vec{x} = \vec{0}$. $\implies$ $E_6 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$.

- Construct matrices P and D.

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then $AP = PD$ and P is invertible, so $P^{-1}AP = D$ and A is diagonalizable.

Linearly independent eigenvectors

Theorem. *If A is an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then the collection of all basis vectors for the eigenspaces $E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_k}$ is linearly independent.*

Proof uses the fact that distinct eigenvalues have eigenvectors that are linearly independent.

Corollary. *If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.*

In our last example, the 3 x 3 matrix A had two distinct eigenvalues:
3 (alg. mult. 2) and 6 (alg. mult. 1).

We know basis for E_3 $\cup$ basis for E_6 is a linearly independent set.

(basis for E_3) can have at most two vectors

(basis for E_6) can only have one vector

For any eigenvalue,
algebraic mult. $\geq$ geometric mult.

Hence, for A to be diagonalizable, we need eigenspace E_3 to be two-dimensional.

Full characterization of diagonalizability

Theorem. *Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then the following are equivalent:*

(1) A is diagonalizable.

(2) Algebraic and geometric multiplicities are equal for each eigenvalue of A.

(3) Union of bases of eigenspaces of A has n vectors.

Invertibility and diagonalizability

Which of the following statements are true?

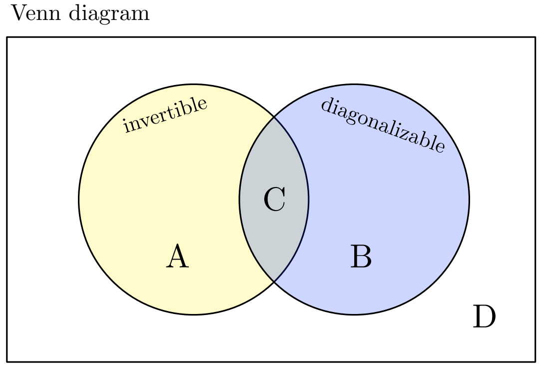
- If A is diagonalizable, then A is invertible.
- If A is invertible, then A is diagonalizable.
- If A is not diagonalizable and not invertible, then A is the matrix of all zeros ($A = 0$).

ALL ARE FALSE!!

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 0 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

Venn diagram



$$C = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Differentiation as a linear transformation

$\mathcal{P}_3 \implies$ set of all polynomials of degree at most 3 with real coefficients
$\mathcal{P}_2 \implies$ set of all polynomials of degree at most 2 with real coefficients

vector spaces

Consider linear transformation $T : \mathcal{P}_3 \to \mathcal{P}_2$ of differentiation:

$$T(p) = p' \quad \text{for polynomials } p \in \mathcal{P}_3.$$

Verify that this is a linear transformation.

In other words,

$$T(ax^3 + bx^2 + cx + d) = 3ax^2 + 2bx + c.$$

What is a basis for $\mathcal{P}_3$?

$$B = \{x^3, x^2, x, 1\}$$

What is a basis for $\mathcal{P}_2$?

$$D = \{x^2, x, 1\}$$

Theorem. *Given a basis $B = \{\vec{v}_1, \ldots, \vec{v}_k\}$ of subspace S, there is a* **unique** *way to express any $\vec{v} \in S$ as a linear combination of basis vectors $\vec{v}_1, \ldots, \vec{v}_k$.*

Encodings and coordinates

Consider linear transformation $T : \mathcal{P}_3 \to \mathcal{P}_2$ of differentiation:

$$T(p) = p' \quad \text{for polynomials } p \in \mathcal{P}_3.$$

$$B = \{x^3, x^2, x, 1\} \qquad D = \{x^2, x, 1\}$$

Then

$$p \in \mathcal{P}_3 \implies p = ax^3 + bx^2 + cx + d \implies [\,p\,]_B = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_B \in \mathbb{R}^4$$

where a, b, c, d are scalars

encoding of p with respect to basis B, or coordinates of v with respect to B

$$q \in \mathcal{P}_2 \implies q = ax^2 + bx + c \implies [\,q\,]_D = \begin{bmatrix} a \\ b \\ c \end{bmatrix}_D \in \mathbb{R}^3$$

where a, b, c are scalars

encoding of q with respect to basis D

Matrix of the linear transformation

Now that we have a way to encode the polynomials, we consider encoding our linear transformation T using a matrix.

$$T : \mathcal{P}_3 \to \mathcal{P}_2 \implies \left[\begin{smallmatrix} 3 \times 4 \text{ matrix} \\ \text{of } T \end{smallmatrix} \right]$$

$B = \{x^3, x^2, x, 1\}$

$$T(x^3) = 3x^2 \quad T(x^2) = 2x \quad T(x) = 1 \quad T(1) = 0$$

To find the matrix of T, we ask ourselves what the linear trans. T does to basis vectors.

$$[3x^2]_D = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}_D \quad [2x]_D = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}_D \quad [1]_D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_D \quad [0]_D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_D$$

$D = \{x^2, x, 1\}$

matrix of T with respect to bases B and D

$$[T]_{D \leftarrow B} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{D \leftarrow B}$$

Using the matrix of T

How do we find the image of a polynomial p under T ?

$$[T]_{D \leftarrow B} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{D \leftarrow B}$$

How do we use T to find the derivative of p ?

Theorem.

$$[T(v)]_D = \left[\quad T \quad \right]_{D \leftarrow B} [v]_B$$

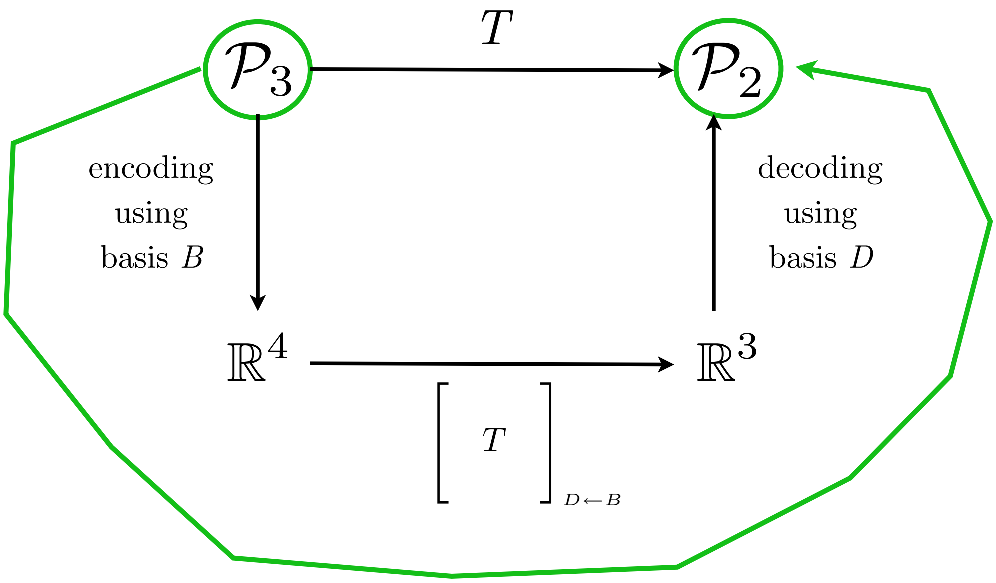
Example: Consider $p(x) = 5x^3 - 3x + 2$. We want to find $T(p(x))$. We have

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{D \leftarrow B} \begin{bmatrix} 5 \\ 0 \\ -3 \\ 2 \end{bmatrix}_B = \begin{bmatrix} 15 \\ 0 \\ -3 \end{bmatrix}_D$$

decode back to polynomial

$$p'(x) = T(p(x)) = 15x^2 - 3$$

Diagram



Change of basis

In our last example, we chose a basis $B = \{x^3, x^2, x, 1\}$ for the vector space $\mathcal{P}_3 \implies$ set of all polynomials of degree at most 3 with real coefficients.

What if we decided we wanted to
use a different basis for this space?

$$B = \{x^3, x^2, x, 1\} \qquad\qquad C = \{(x+1)^3, x^2, x+1, x-1\}$$

For example,

$$[\, x^3 \,]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}_B \quad \text{and} \quad [\, x^3 \,]_C = \begin{bmatrix} 1 \\ -3 \\ -2 \\ -1 \end{bmatrix}_C$$

Is it possible to find $[\, x^3 \,]_C$ from $[\, x^3 \,]_B$ using matrix multiplication?

Yes!!

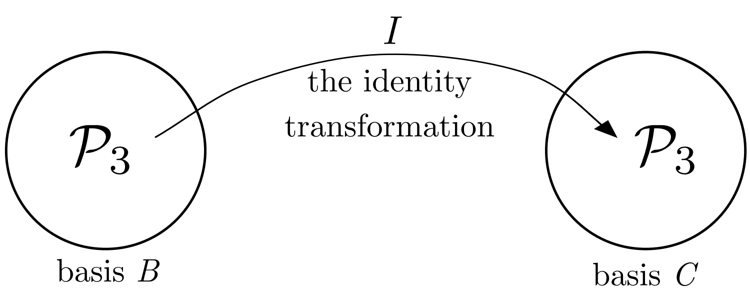
Change of basis

What is the relationship between encodings of polynomials with respect to B and those with respect to C?

$$B = \{x^3, x^2, x, 1\} \qquad C = \{(x+1)^3, x^2, x+1, x-1\}$$

For example,

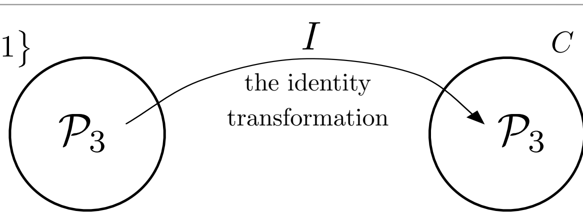
$$[\,x^3\,]_B = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}_B \quad \text{and} \quad [\,x^3\,]_C = \begin{bmatrix}1\\-3\\-2\\-1\end{bmatrix}_C$$



Computing change-of-basis matrix

$B = \{x^3, x^2, x, 1\}$

$C = \{(x+1)^3, x^2, x+1, x-1\}$



To find the matrix of I, we ask ourselves what the linear trans. I does to basis vectors.

$$I(x^3) = x^3 \qquad I(x^2) = x^2 \qquad I(x) = x \qquad I(1) = 1$$

$$[\,x^3\,]_C = \begin{bmatrix}1\\-3\\-2\\-1\end{bmatrix}_C \quad [\,x^2\,]_C = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}_C \quad [\,x\,]_C = \begin{bmatrix}0\\0\\\frac{1}{2}\\\frac{1}{2}\end{bmatrix}_C \quad [\,1\,]_C = \begin{bmatrix}0\\0\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}_C$$

change-of-basis matrix from B to C

$$[\,I\,]_{C\leftarrow B} = \begin{bmatrix}1 & 0 & 0 & 0\\-3 & 1 & 0 & 0\\-2 & 0 & \frac{1}{2} & \frac{1}{2}\\-1 & 0 & \frac{1}{2} & -\frac{1}{2}\end{bmatrix}_{C\leftarrow B}$$