ANALYSIS OF N-CARD LE HER

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Abstract. We present a complete solution to a card game with historical origins. Our analysis exploits convexity properties in the payoff matrix, allowing this discrete game to be resolved by continuous methods.

In this paper, we analyze a variant of the card-game Le Her, which has a long history in the mathematical literature (cf. Section 18.6 of [7]). The authentic two-player 52-card version is reported and solved by Dresher [3, 4], with some anticipation by R.A. Fisher [6]. Todhunter [10] describes efforts at its solution by N. Bernoulli and Montmort; retrospectively, their lack of the "mixed strategy" concept can be recognized as crucial. The present version, formulated by Karlin ([9], p. 100) who poses the case $N = 5$ as a problem, involves a single-suit deck of $N \geq 3$ cards with respective face-values $1, 2, \ldots, N$. Let X, Y, Z denote the top three cards, all face down, after a randomizing shuffle of the deck. Cards X and Y are dealt to Players 1 and 2 respectively, leaving Z on top. Each player inspects his or her own card; thus Player 1 knows X but not Y or Z, while Player 2 knows Y but not X or Z. (Our notation will slur the distinction between a random variable and its realization, without real risk of confusion.)

Player 1 moves first. He can either *keep* X, or else elect a *swap* of cards with Player 2. In the latter case, both players inspect their new cards, and therefore know both X and Y (but not Z). Next, it is Player 2's turn to move. She can either *keep* her current card, or else elect to *swap* that card for Z. That concludes play: the player holding the higher card *wins* one unit from the opponent.

The pure strategies for Player 1 are associated one-to-one with the subsets S of $\{1, 2, \ldots, N\}$; playing strategy S, Player 1 keeps X if $X \in S$, and otherwise swaps X for Y. Note that if Player 1 swaps, then Player 2 will know that she holds X while Player 1 is holding Y, and will therefore surely keep X if $X > Y$, and surely swap X for Z if $X < Y$, the swap "succeeding" iff $Z > Y$. Thus a pure strategy for Player 2 need specify that player's action only when Player 1 keeps his card. Those strategies are associated one-to-one with subsets T of $\{1, 2, \ldots, N\}$; playing strategy T after a "keep" by Player 1, Player 2 keeps Y if $Y \in T$, and otherwise swaps Y for Z.

The joint distribution of (X, Y) is of course given by $P(X, Y) = p = \frac{1}{N(N-1)}$ if $X \neq Y$, $P(X, Y) = 0$ if $X = Y$. Let $P(S, T|X, Y)$ denote the probability of a win by Player 1 when respective strategies S and T are employed, conditional on (X, Y); the corresponding *unconditional* probability is given by

$$P(S,T) = \sum_{X,Y} P(S,T|X,Y)P(X,Y) = p \sum_{X \neq Y} P(S,T|X,Y). \tag{1}$$

The entries of the payoff matrix are given by $1 \cdot P(S,T) + (-1) \cdot [1 - P(S,T)]$, but an equivalent strategic analysis results if this payoff function is replaced by $P(S,T)$, and we will do so.

The preceding description yields a $2^N \times 2^N$ matrix game. But this size can clearly be reduced; for instance, it would be foolish to swap away the highest possible card N, so that rational play requires $N \in S \cap T$. Intuitively, Players 1 and 2 can restrict themselves to "gapless" strategies. That is, Player 1 has a critical number $s \geq 2$ where he will keep exactly those cards greater than or equal to s, and Player 2 has a critical number $t \geq 2$ where if Player 1 keeps his card, she will keep just those cards that are greater than or equal to t. For brevity we omit the confirmation (in [1]) of this intuition, i.e., the formal proof that any strategy that keeps k cards is dominated by the strategy that keeps the k cards of highest value. We let $S_s = \{X : X \geq s\}$ and $T_t = \{Y : Y \geq t\}$. Thus our matrix game can be reduced to the $(N-1) \times (N-1)$ payoff matrix $A = [a(s,t)]$, where $a(s,t) = P(S_s, T_t)$ for $2 \leq s, t \leq N$. Our goal is to establish a radical further reduction, to an explicitly-identified submatrix game of size at most only 2×2, whose solution is therefore given by standard formulas.

Next we provide a formula for $a(s,t)$. The formula depends on the sign of $s - t$. As preparation, we define for integral $s, t \in [2, N]$,

$$\begin{aligned}
(2) h(s,t) &= (N-t)(N+1-t) + (N-1)(s+t-2), \\
(3) f(s,t) &= h(s,t) - (s-1)(s-2)(s+3t-12)/3(N-2), \\
(4) g(s,t) &= h(s,t) - (s-2)(s-3)(s+3t-4)/3(N-2) - (s-t)(s-t-1),
\end{aligned}$$

and observe that $f(s,t) = g(s,t)$ when $s = t$.

Lemma 1. *For $s \leq t$, $2a(s,t)/p = f(s,t)$.*

Proof. There are three cases to consider in evaluating (1) for $S = S_s$ and $T = T_t$. If $X < s \leq t$ then Player 1 swaps, and wins iff $X < Y$ and $Z < Y$. If $s \leq X < t$ then Player 1 keeps X; since Player 2 wins if he keeps (i.e., if $Y \geq t$), Player 1 wins iff $Y < t$ and $Z < X$, the latter corresponding to $X - 1$ possibilities for Z if $X < Y$ but to only $X - 2$ (the value Y is ruled out) if $Y < X$. Finally, if $X \geq t$ then Player 1 wins iff either $X > Y \geq t$ or $Y < t \leq X$ and $Z < X$ (the value Y is ruled out for Z). These cases yield three groups of terms in the evaluation of (1):

$$\begin{aligned}
\frac{1}{p} a(s,t) &= \sum_{X<s} \sum_{Y>X} \frac{Y-2}{N-2} \\
&+ \sum_{s \leq X < t} \sum_{X<Y<t} \frac{X-1}{N-2} + \sum_{s \leq X < t} \sum_{Y<X} \frac{X-2}{N-2}. \\
&+ \sum_{X \geq t} \sum_{t \leq Y < X} 1 + \sum_{X \geq t} \sum_{Y<t} \frac{X-2}{N-2}.
\end{aligned}$$

Heroic algebra, and the formula for the sum of the first $s-1$ perfect squares, yield the stated result. □

Lemma 2. *For $s \geq t$, $2a(s,t)/p = g(s,t)$.*

Proof. Now there are two cases to consider. If $X < s$ then Player 1 swaps, and wins iff $X < Y$ and $Z < Y$. And if $X \geq s \geq t$ then Player 1 keeps X, winning iff either $Y < t$ and $Z < X$, or $X > Y \geq t$. These cases yield two groups of terms in the evaluation of (1):

$$\begin{aligned} \frac{1}{p}a(s,t) &= \sum_{X<s}\sum_{Y>X}\frac{Y-2}{N-2} \\ &+ \sum_{X\geq s}\sum_{Y<t}\frac{X-2}{N-2} + \sum_{X\geq s}\sum_{X>Y\geq t} 1, \end{aligned}$$

from which algebraic manipulation leads to the stated result. □

The payoff function $a(s,t)$ has some desirable properties. By straightforward algebra, one can show [1] that the columns of the payoff matrix A are *discrete concave* and the rows of A are *discrete convex*. That is,

Lemma 3. *For each* $2 \leq t \leq N$*,* $a(s+1,t) - a(s,t)$ *is nonincreasing in* $s \geq 2$*.*

Lemma 4. *For each* $2 \leq s \leq N$*,* $a(s,t+1) - a(s,t)$ *is nondecreasing in* $t \geq 2$*.*

Howard [8] proves that in a game that satisfies the concavity condition of Lemma 3 Player 1 has an optimal mixed strategy that mixes at most two *consecutive* pure strategies. Analogous results apply for Player 2 when the convexity condition of Lemma 4 occurs. (An alternative treatment, whose preview in [1] was apparently the stimulus for [8], occurs in [2].) Since the only difference between consecutive strategies S_s and S_{s+1} is how they treat card s, then the above results give us:

Theorem 1. *In our variant of* N*-card Le Her, Player 1 has a critical card* s *such that he will always swap cards below* s*, always keep cards above* s *and will keep or swap card* s *according to a mixed strategy. Likewise, Player 2 has a critical card* t *such that when Player 1 keeps his card, she will always swap cards below* t*, always keep cards above* t *and will keep or swap card* t *according to a mixed strategy.*

To determine *which* consecutive strategies are optimal, suppose that $a(s,t)$ can be interpolated by a function $A(s,t)$ defined on the real domain $[2,N] \times [2,N]$, where for fixed t, A is a concave function of s. By [5], such a game has an optimal *pure* strategy s^*. The authors prove in [2] that in the discrete version of such a game, Player 1 has an optimal strategy which mixes at most the pure strategies $\lfloor s^* \rfloor$ and $\lceil s^* \rceil$. Likewise, if A is a convex function of t, then the continuous game will have a pure optimal strategy t^*, and the original game will optimally mix on pure strategies $\lfloor t^* \rfloor$ and $\lceil t^* \rceil$.

The right-hand sides of equations (2–4) yield extensions (H,F,G) of the respective functions (h,f,g) from integer to continuous variables $(s,t) \in [2,N]$, with $F = G$ when $s = t$. By Lemmas 1 and 2, a "natural" extension $A(s,t)$ of $a(s,t)$ to a continuous-game payoff function is given by

$$\frac{2}{p}A(s,t) = \begin{cases} F(s,t) & \text{for } s \leq t, \\ G(s,t) & \text{for } s \geq t. \end{cases} \tag{5}$$

To solve the game as proposed in the previous paragraph, we need to verify that $A(s,t)$ – or equivalently, $\frac{2}{p}A(s,t)$ – is concave in s for fixed t. We first observe by straightforward differentiation that

$$\partial^2 G/\partial s^2 = -2(N+s+t-5)/(N-2),$$

which is non-positive (as desired) since $s,t \in [2,N]$, and that

$$\partial^2 F/\partial s^2 = -2(s+t-5)/(N-2),$$

which has the desired sign except in the upper left corner (defined by $s+t<5$) of the square $[2,N]\times[2,N]$. In view of (5), it is also necessary for concavity (in s) to check that $\partial F/\partial s \geq \partial G/\partial s$ when $s=t$. This condition reduces to the explicit form

$$t \geq \Theta =_{\text{def}} (N+2)/2, \tag{6}$$

leaving the subinterval $[2,\Theta)$ to be dealt with. The next lemma shows that this initial subinterval of $[2,N]$ can be eliminated by a suitable domination argument on the matrix game. In what follows, we use the floor and ceiling symbols $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote the greatest integer less than or equal to x and the least integer greater than or equal to x, respectively.

Lemma 5. *For integral $s \in [2,N]$, $a(s,t) \geq a(s,\lceil\Theta\rceil)$ holds for all integral $t \in [2,\lceil\Theta\rceil]$.*

Proof. For continuous (s,t) with $2 \leq t \leq s \leq N$, we have

$$2p^{-1}\partial A/\partial t = \partial G/\partial t = -(N-s)(N-s+1)/(N-2) \leq 0, \tag{7}$$

so that $A(s,t)$ is nonincreasing in t. And if $s \leq t$, then

$$2p^{-1}\partial A/\partial t = \partial F/\partial t = -\{(N+2-2t)+(s-1)(s-2)/(N-2)\}, \tag{8}$$

yielding the same conclusion if also $t \leq \Theta$. Thus $a(s,t)$ is nonincreasing in t for $t \leq \Theta$, yielding the desired result when N is even so that Θ is integral. And if N is odd, then $a(s,t)$ is nonincreasing in t for $t \leq \lfloor\Theta\rfloor = (N+1)/2$; the remaining desired conclusion $a(s,\lfloor\Theta\rfloor) \geq a(s,\lceil\Theta\rceil)$ follows from (7) if $\lceil\Theta\rceil \leq s$, while if integral $s < \lceil\Theta\rceil$ (i.e., $s \leq \lfloor\Theta\rfloor$) then integration over $[\lfloor\Theta\rfloor,\lceil\Theta\rceil] = [\Theta-\frac{1}{2},\Theta+\frac{1}{2}]$ of the expression in (8) implies the result via the conclusion

$$2p^{-1}[a(s,\lfloor\Theta\rfloor)-a(s,\lceil\Theta\rceil)] = -(s-1)(s-2)/(N-2) \leq 0.$$

□

It follows from Lemma 5 that Player 2's pure strategies in the matrix game can be restricted by $t \geq \lceil\Theta\rceil$, so that the same can be done in the continuous extension. We may assume that $N \geq 4$ (since if $N=3$ the matrix game is already 2×2), so that the last restriction implies $t \geq 3$, which in junction with $s \geq 2$ rules out the troublesome corner $s+t<5$. Thus the concavity-in-s property has been established. We note in passing the following intuitively plausible interpretation, in the matrix game, of the domination-enforced condition (6): If Player 1 has kept his card, then Player 2 should swap any card that is not above average.

We continue to assume $N \geq 4$, and now know that the continuous game with payoff function $A(s,t)$ restricted to the rectangle $[2,N]\times[\lceil\Theta\rceil,N]$ has some optimal pure strategy s^* for Player 1, and that in the matrix game Player 1 has an optimal strategy which mixes at most the consecutive rows $\lfloor s^*\rfloor$ and $\lceil s^*\rceil$. To identify these rows, we proceed to determine s^*. By the "maximin" definition of an optimal strategy for Player 1, s^* is characterized by maximizing, over $[2,N]$, the function

$$\mu(s) = \min\{A(s,t) : \lceil\Theta\rceil \leq t \leq N\}, \tag{9}$$

i.e., $\mu(s) = A(s,t^*(s))$ where $t^*(s)$ minimizes $A(s,t)$ over $[\lceil\Theta\rceil,N]$. By (7), we have $t^*(s) \geq s$ if $s<N$ and can take $t^*(s) \geq s$ if $s=N$, so that

$$\mu(s) = F(s,t^*(s)) \tag{10}$$

where $t^*(s)$ minimizes $F(s,t)$ over $[\max(s,\lceil\Theta\rceil),N]$.

To determine $t^*(s)$, we use (8) to equate $\partial F/\partial t$ to 0, obtaining the t-value

$$\tau^*(s) = \Theta + (s-1)(s-2)/2(N-2).$$

It follows from (8) that $t^*(s)$ is given by $\tau^*(s)$ if the latter lies in the interval $[\max(s, \lceil\Theta\rceil), N]$. Analyzing the conditions for membership of $\tau^*(s)$ in this interval, we find that $\tau^*(s) \geq s$ is equivalent to $(N-s)^2 + (s-2) \geq 0$, which is true. Next, $\tau^*(s) \leq N$ is equivalent to $(s-1)(s-2) \leq (N-2)^2$, which is true for $s \leq N-1$ but *not* for $s = N$. Finally, since $s \geq 2$, $\tau^*(s) \geq \lceil\Theta\rceil$ is true when N is even (so that Θ is integer), but for odd N it is equivalent to

$$(s-1)(s-2) \geq N-2,$$

which *fails* for sufficiently small s.

However, one can show (see [1]) that the "troublesome cases" mentioned above need never occur in an optimal solution. That is, without loss of optimality, Player 1 can restrict himself to

$$s \leq N-1 \text{ and } (s-1)(s-2) \geq N-2. \tag{11}$$

These conclusions follow (respectively) from the next two additional domination results about the matrix game, whose proofs (in [1]) are again omitted for brevity:

Lemma 6. *For integral* $t \in [2, N]$, $a(N,t) \leq a(N-1,t)$.

Lemma 7. *For integral* $t \in [\lceil\Theta\rceil, N]$ *and integral* s *with* $(s-1)(s-2) < N-2$, $a(s+1,t) > a(s,t)$.

We have now justified equating $t^*(s)$ to $\tau^*(s)$, i.e.,

$$t^*(s) = \Theta + (s-1)(s-2)/2(N-2). \tag{12}$$

Substitution of (12) into (10), and differentiation, yield for $-6(N-2)^2 d\mu/ds$ the expression

$$\phi(s) = 6s^3 + (6N-39)s^2 + (6N^2-60N+135)s - (6N^3-21N^2-28N+110). \tag{13}$$

Its derivative is a quadratic function whose discriminant $-36(8N^2-68N+110)$, is negative (hence $\phi(s)$ is increasing) for $N \geq 7$, where it is easily verified that $\phi(N-1) > 0 > \phi(\Theta)$. Thus for $N \geq 7$ the unique real root of $d\mu/ds = 0$, is interior to the interval $[\Theta, N-1]$, hence satisfies (11), so that s^* can be calculated as the real root of $\phi(s) = 0$. As for the remaining small values of N, according to (11) Player 1's pure strategies can be confined to $s = 3$ if $N = 4$, to $s = 4$ if $N = 5$, and to the consecutive pair $s \in \{4, 5\}$ if $N = 6$. In these three cases the restriction $t \geq \lceil\Theta\rceil$ translates into $t \in \{3, 4\}$, $t \in \{4, 5\}$ and $t \in \{4, 5, 6\}$ respectively; in the last of these the third column of the 2×3 submatrix coincides with the second, permitting reduction to a 2×2 game. So for what follows, we can and will assume $N \geq 7$.

We have showed that the $(N-1) \times (N-1)$ matrix game can be reduced to a subgame involving the last $N - \lceil\Theta\rceil + 1$ columns and at most a consecutive pair $(\lfloor s^*\rfloor, \lceil s^*\rceil)$ of rows, and a procedure for determining this pair has been given. As noted after Lemma 4, we are also assured that in principle this subgame can be reduced further to a sub-subgame of dimensions at most 2×2 involving consecutive columns $(\lfloor t^{**}\rfloor, \lceil t^{**}\rceil)$. For given N it seems brute-force practical to proceed by successive solution of 2×2 sub-subgames involving consecutive columns, retaining the solution with the smallest payoff value. However, it would be more elegant to mirror the preceding analysis from Player 2's viewpoint, giving a "semi-closed" recipe for t^{**}.

Such an attempt would naturally begin by verifying that for fixed s, $A(s,t)$ is convex in t. We find by straightforward differentiation that

$$\partial^2 F/\partial t^2 = 2,\ \partial^2 G/\partial t^2 = 0 \tag{14}$$

which by (5) assures convexity over the separate t-intervals $(s,N]$ and $[2,s)$. But in view of (5), it is also necessary to check that $\partial F/\partial t \geq \partial G/\partial t$ when $t = s$. This condition reduces to the explicit form $s \leq \Theta$, whereas we showed above (second sentence after (13) that $s^* > \Theta$. So our mirror must be blurred by an additional line of argument.

Theorem 2. *For $N \geq 7$, optimal mixed strategies for our variant of Le Her can be obtained by solving the 2×2 subgame involving only rows $\lfloor s^* \rfloor$ and $\lceil s^* \rceil$, where, s^* is the real zero of the cubic $\phi(s)$ defined by (13), and only columns $\lfloor t^{**} \rfloor$ and $\lceil t^{**} \rceil$, where $t^{**} = \max(t^*(s^*), \lceil s^* \rceil)$ as defined by (12) and (6).*

Proof. It has already been proved that attention can be restricted to the rows $\lfloor s^* \rfloor$ and $\lceil s^* \rceil$, and to columns $t \geq \lceil \Theta \rceil$. We first show that the latter restriction can be tightened to $t \geq \max(\lceil \Theta \rceil, \lfloor s^* \rfloor)$. (Since the material following (13) yields $s^* > \Theta$, this tightening might be a strict one.) For this purpose note that by (7), for integer $t \leq \lfloor s^* \rfloor \leq \lceil s^* \rceil$, we have

$$a(\lfloor s^* \rfloor, t) \geq a(\lfloor s^* \rfloor, \lfloor s^* \rfloor),\ a(\lceil s^* \rceil, t) \geq a(\lceil s^* \rceil, \lfloor s^* \rfloor),$$

so that in the 2-rowed matrix subgame column t is dominated by column $\lfloor s^* \rfloor$ and can therefore be deleted if $t < \lfloor s^* \rfloor$.

We next show that if s^* is non-integer and the surviving matrix subgame still contains column $\lfloor s^* \rfloor$, then that column is dominated by column $\lceil s^* \rceil$ and can therefore be deleted. For this we must demonstrate

$$a(\lfloor s^* \rfloor, \lfloor s^* \rfloor) \geq a(\lfloor s^* \rfloor, \lceil s^* \rceil),\ a(\lceil s^* \rceil, \lfloor s^* \rfloor) \geq a(\lceil s^* \rceil, \lceil s^* \rceil).$$

The second assertion with $t \leq s$ on both sides, is a consequence of (7). The first assertion, since $s \leq t$ on both sides, is by Lemma 1 an instance of the relation $f(s,s) \geq f(s,s+1)$. Using (2) and (3), we find this relation to take the explicit form

$$(N + 1 - 2s) + (s-1)(s-2)/(N-2) \geq 0,$$

which is readily verified to hold for integral s, failing only in $(N-1,N)$.

Now the matrix subgame is restricted to rows $\lfloor s^* \rfloor$ and $\lceil s^* \rceil$, and to columns $t \geq \max(\lceil \Theta \rceil, \lceil s^* \rceil) = \lceil s^* \rceil$. Our continuous extension can therefore be restricted to the corresponding strip in the square $[2,N] \times [2,N]$, throughout which $s \leq t$, so that $A(s,t) = \frac{p}{2}F(s,t)$. The first part of (14) now establishes strict convexity of $A(s,t)$ in t for fixed s throughout the strip. Thus the restricted continuous game has a pure optimal strategy t^{**} for Player 2, and the remarks following Theorem 1 assure that the matrix game can be further limited to columns $\lfloor t^{**} \rfloor$ and $\lceil t^{**} \rceil$. Since s^* remains optimal for Player 1 in the restricted continuous game, t^{**} can be identified as a minimizer of $A(s^*,t) = \frac{p}{2}F(s^*,t)$ over $[\lceil s^* \rceil, N]$. Thus t^{**} coincides with $t^*(s^*) = \tau^*(s^*)$ if the latter is $\geq \lceil s^* \rceil$; if not, the convexity (in t) of $A(s^*,t)$ identifies t^{**} as the nearest feasible point to the "relaxed minimizer" $t^*(s^*)$, i.e., $t^{**} = \lceil s^* \rceil$. $\square$

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References

[1] Benjamin, A.T., and A.J. Goldman, "Le Her", preprint, (1992).

[2] Benjamin, A.T., and A.J. Goldman, "Localization of Optimal Strategies in Certain Games", *Naval Research Logistics*, **41**, 669–676, (1994).

[3] Dresher, M., "Games of Strategy", *Math. Magazine*, **25**, 93–99 (1951).

[4] Dresher, M., *Games of Strategy: Theory and Applications*, Prentice-Hall, Englewood Cliffs, N.J., 1961.

[5] Fan, K., "Minimax Theorems", *Proceedings of the National Academy of Science*, **39**, 42–47 (1953).

[6] Fisher, R.A., "Randomisation and an Old Enigma of Card Play", *Math. Gazette*, **18**, 294–297 (1934).

[7] Hald, A., *A History of Probability and Statistics and Their Applications before 1750*, John Wiley & Sons, New York, 1990.

[8] Howard, J.V., "A Geometrical Method of Solving Certain Games", *Naval Research Logistics* **41**, 133–136 (1994).

[9] Karlin, S., *Mathematical Methods and Theory in Games, Programming and Economics*, Vol. I, Addison-Wesley, Reading, Mass., 1959.

[10] Todhunter, I., *A History of the Mathematical Theory of Probability*, Chelsea, New York, 1949 (Reprint).

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