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ANOTHER WAY OF COUNTING N^N*

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Abstract. An original combinatorial proof of a combinatorial identity is presented.

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1. Introduction. We provide a combinatorial proof that

$$N^N = \sum_{\substack{0 \le k_1 \le 1 \\ 0 \le k_1 + k_2 \le 2 \\ \vdots \\ 0 \le k_1 + k_2 + \cdots + k_{N-1} \le N-1}} \cdots \sum \frac{N!}{k_1! k_2! \cdots k_{N-1}!}.$$

2. Sequence graphs and cycle sets. We count the number of sequences of N integers between 1 and N, which we call N-sequences, in two ways, which give us the left- and right-hand sides of our equation, respectively. Counting in the obvious way, there are N choices for each integer, yielding N^N total N-sequences. Next, after setting up some apparatus, we answer the same question in a way that yields the summation.

As is done in [1] and [2], we define an *N-sequence graph* (*N-graph*) as a directed graph of N nodes, where each node has out-degree 1. We can set up a one-to-one correspondence between N-sequences and N-graphs as follows: Let the (directed) edge originating at node i of a N-graph terminate at node j, where j is the ith integer of the corresponding N-sequence. For example, the 12-sequence 433744657575 corresponds to the 12-graph of Fig. 1. This one-to-one correspondence allows us to count N-graphs instead of N-sequences.

To attain the right-hand side of our identity, we associate with each N-graph a unique sequence $k_1, k_2, \cdots, k_{N-1}$, which meets the conditions of the summation. We then show that the number of N-graphs associated with $k_1, k_2, \cdots, k_{N-1}$ is $N!/k_1!k_2! \cdots k_{N-1}!$.

To that end, we define the *cycle set* C of an N-graph to be the set of all nodes in the graph which are contained in at least one cycle. (Since our graph is finite, and each node has out-degree 1, it is clear that C is nonempty.) Note that, by construction, all edges originating at nodes of C point to nodes of C, and no two of these edges terminate at the same node. In our example, $C = \{3, 4, 6, 7\}$.

3. Ordered forests. An N-graph can be reduced to a forest by removing the edges originating at nodes contained in C, and letting the nodes in C serve as roots. The forest generated by doing this to the 12-graph of Fig. 1 is shown in Fig. 2. To remove ambiguity, we then rearrange the forest into what we will call an *ordered forest* by placing the roots in ascending order from left to right, and doing the same for all *sibling sets* (i.e., children of the same node). (See Fig. 3.)

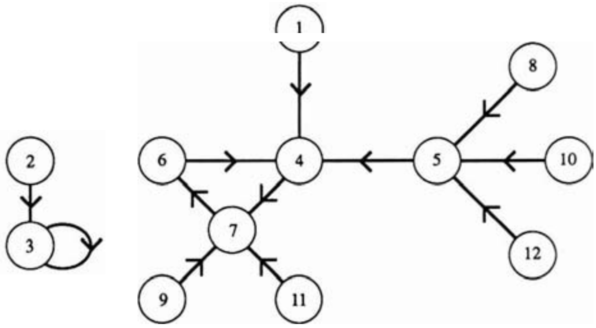

FIG. 1. *The* 12-*graph of the* 12-*sequence* 433744657575.

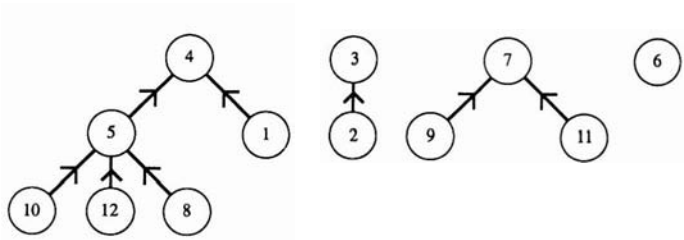

FIG. 2. *A forest from* 433744657575.

Next, we label each tree from left to right in *postorder*. Starting with the root node of a tree, the postorder labeling process may be defined recursively as follows: postorder the subtrees whose roots are the nodes that point to the current root, going from left to right among these subtrees, then label the current root. (See [3].) We let the first postorder label be zero. In Fig. 3, we have written the postorder label alongside each node. Note that, by construction, the label assigned to each node must be greater than the labels of its descendant nodes.

We can now uniquely determine the sequence $k_1, k_2, \cdots, k_{N-1}$ from our ordered forest by letting k_j be the number of children of the node with postorder label j. Note that k_0 must always be zero. In Fig. 3, k_1 through k_{11} are equal to 1, 0, 0, 0, 0, 3, 2, 0, 0, 0, 2, respectively. Note that the condition $0 \le k_1 + k_2 + \cdots + k_j \le j$ is met for all j between 1 and 11. This is true in general: $k_1 + k_2 + \cdots + k_j$ is equal to the number of children possessed by nodes 0 through j, all of which must have postorder. labels less than j.

It remains to show that there are $N!/k_1!k_2! \cdots k_{N-1}!$ N-graphs with k-sequence $k_1, k_2, \ldots, k_{N-1}$. In [3] it is proved that every *unlabeled* rooted forest is characterized by its postorder degree sequence. Hence our k-sequence completely determines the shape of the ordered forest. For example, any unlabeled forest with k-sequence 10000320002 must look like Fig. 4.

Since the elements of C and the sibling sets (circled in Fig. 4) must be written in ascending order, we can convert the unlabeled forest of Fig. 4 into a labeled ordered

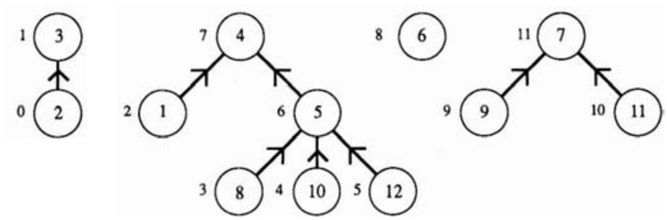

FIG. 3. *The ordered forest of* 433744657575 *with postorder labels.*

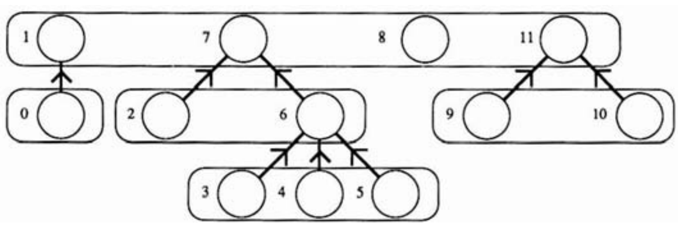

FIG. 4. *Unlabeled graph for k-sequence* 10000320002.

forest in $12!/1!3!2!2!4!$ ways.

In general, any unlabeled forest with k-sequence $k_1, k_2, \cdots, k_{N-1}$ can be labeled $N!/k_1!k_2!\cdots k_{N-1}!|\mathcal{C}|!$ ways. Finally, each labeled ordered forest with root set $\mathcal{C}$ gives rise to $|\mathcal{C}|!$ N-graphs because the edges leaving the nodes of $\mathcal{C}$ must enter different nodes of $\mathcal{C}$.

Therefore each k-sequence corresponds to $N!/k_1!k_2!\cdots k_{N-1}!$ N-graphs, as asserted. Since each sum counts the correct number of N-graphs, we have finished our second count and have established the identity.

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