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CHALLENGING KNIGHT'S TOURS

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Arthur T. Benjamin and Sam K. Miller

Over the centuries, chess enthusiasts have enjoyed solving a problem known as the *knight's tour*. In this problem, we are given a standard eight-by-eight chessboard and a knight that is placed on an arbitrary square. Our goal is to move the knight 63 times so that it visits every square exactly once. (A knight can make L-shaped jumps—two squares in one direction and one square in a perpendicular direction.)

Variants of this problem add to the challenge. For example, we could require that the last visited square is a knight's move away from the starting point. An even harder version specifies a starting square *and* an ending square of opposite color; we call this the *challenging knight's tour*. Note that a knight always jumps to a square of the opposite color; hence, it is impossible for the last square to be the same color as the first one.

In this article, we give a constructive proof that all start- and end-square combinations of the challenging knight's tour have a solution. Although computer programs have proved this result by finding tours for all possible start and end points, we present a short proof that provides insights to the problem's structure (Michael Dupuis and Stan Wagon, "Laceable Knights," *Ars Math. Contemp.* 9 [2015]: 115–124).

	a	b	c	d	e	f	g	h
8	*d*	*s*	*S*	*D*	*d*	*s*	*S*	*D*
7	*S*	*D*	*d*	*s*	*S*	*D*	*d*	*s*
6	*s*	*d*	*D*	*S*	*s*	*d*	*D*	*S*
5	*D*	*S*	*s*	*d*	*D*	*S*	*s*	*d*
4	*d*	*s*	*S*	*D*	*d*	*s*	*S*	*D*
3	*S*	*D*	*d*	*s*	*S*	*D*	*d*	*s*
2	*s*	*d*	*D*	*S*	*s*	*d*	*D*	*S*
1	*D*	*S*	*s*	*d*	*D*	*S*	*s*	*d*

Figure 1. The vertices of the knight's tour graph can be partitioned into four sets and labeled *d*, *D*, *s*, and *S*, denoting left diamond, right diamond, left square, and right square, respectively.

The Knight's Tour Graph

For a mathematician, the knight's tour is a problem in graph theory. The chessboard graph G has 64 vertices, each representing a square on the board. There is an edge connecting vertices x and y if a knight can jump from square x to square y in a single move; we say that squares x and y are *adjacent* in G. Each vertex is black or white, depending on the color of its square.

Thus, the challenging knight's tour problem can be stated as follows: Given two vertices in G of opposite colors (called o and e for origin and endpoint), find a path in G from o to e that visits every vertex exactly once; this is called a *Hamiltonian path*.

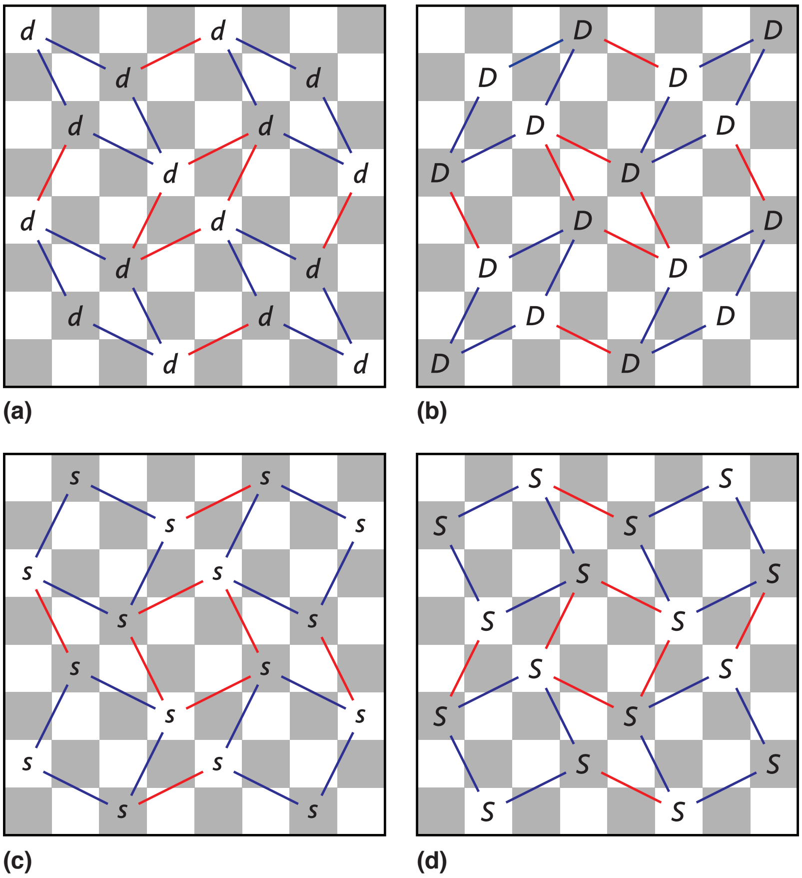

Figure 2. (a) The left diamond, (b) right diamond, (c) left square, and (d) right square systems.

Each square in the chessboard is identified by its rank (numbered row) and file (lettered column). Thus, we may refer to a square and its corresponding vertex by its rank and file address; for instance, the lower left and upper right squares are a1 and h8, respectively.

The graph G has 168 edges, but we can partition the 64 vertices into four sets of size 16 that illuminate its structure. We label each vertex with one of four symbols, d, D, s, or S, as shown in figure 1. Vertices with label d represent squares in the *left diamond system*. Notice that in the subgraph shown in figure 2a (called the *induced subgraph*) each quadrant has four vertices joined by blue edges in the shape of a diamond that leans to the left (mnemonic: left diamond = lowercase d).

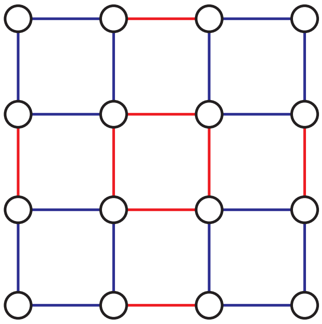

Figure 3, above. All four induced subgraphs are isomorphic to $G_{4,4}$.

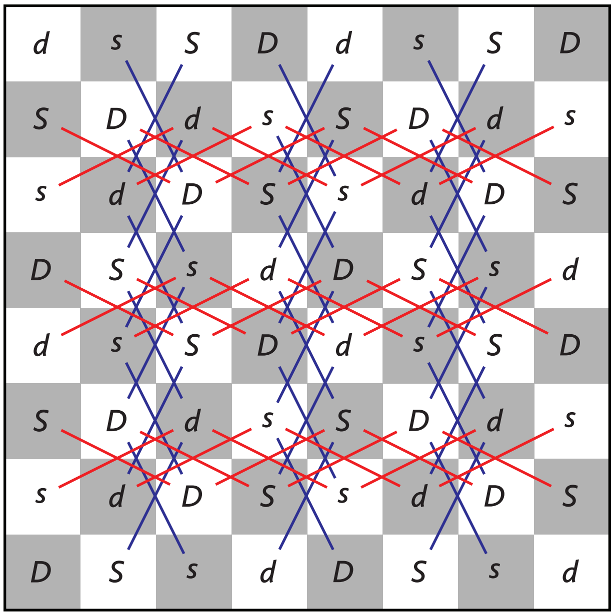

Figure 4, right. The *Sd* and *sD* edges form a (blue) shoelace pattern, and the *sd* and *SD* edges form a (red) shoelace pattern.

Vertices with label D belong to the *right diamond system*, with the induced subgraph shown in figure 2b. Similarly, vertices with labels s or S belong to the *left square* or *right square system*, illustrated in figures 2c and 2d, respectively.

We say that the labels d and D are of the same *type*, as are s and S.

Notice that the 16 vertices and 24 edges of the d-graph can be straightened into the four-by-four graph $G_{4,4}$, shown in figure 3. For instance, the lower left vertex of $G_{4,4}$ represents the square b2. In fact, all four of the induced subgraphs have the same underlying structure; we say they are each *isomorphic* to $G_{4,4}$.

What about the other edges of G? Notice that there are no edges that connect two different systems of the same type; that is, there are no dD or sS edges. In figure 4, we see that the 36 (blue) sD and Sd edges form a shoelace pattern, as do the 36 (red) sd and SD edges. Combining all 168 edges of figures 2a–d, and 4 gives us the knight's tour graph G (not pictured).

Strategy

Once we are aware of the four systems, it is easy to solve the basic knight's tour problem, in which we are given the initial vertex o but have the freedom to end anywhere we want.

Suppose o is in the d system, say in the lower left quadrant of the board. Jump to the three other vertices in that quadrant with label d (being careful not to end on the corner vertex b2). Then jump to an adjacent quadrant, landing on a d vertex, and repeat the process. Thus, in 15 jumps, we can visit all 16 d vertices. Next, jump to a vertex with a different label, say s, and cover the s-system. Then cover the D-system, and end with the S-system. Go ahead, try it!

To solve the challenging knight's tour where the first and last vertices are given, we use a pseudo-algorithm devised by Michael Daniels ("Learn to Master the Knight's Tour," *mindmagician.org/tourhelp.aspx*).

Our primary observation is that we can use the four *middle vertices* of each system—the vertices with ranks 3, 4, 5, and 6 and files c, d, e, and f—to jump from one system to either system of opposite type (see figure 4). In fact, each of these vertices is adjacent to two vertices of the opposite type. For example, the d-graph has middle vertices c3, d5, e4, and f6. And from c3, say, we can reach two vertices of type s (a2 and e2) and two of type S (b1 and b5).

As with the basic knight's tour problem, our general strategy is to traverse the board one system at a time. However, in some cases, we will need to be clever in our ordering.

We now come to our first important lemma, which will be the basis for our constructive proof. A chessboard graph is *traversable* if we can find a Hamiltonian path from any starting vertex to any ending vertex of the opposite color. We call such a path a *traversal*.

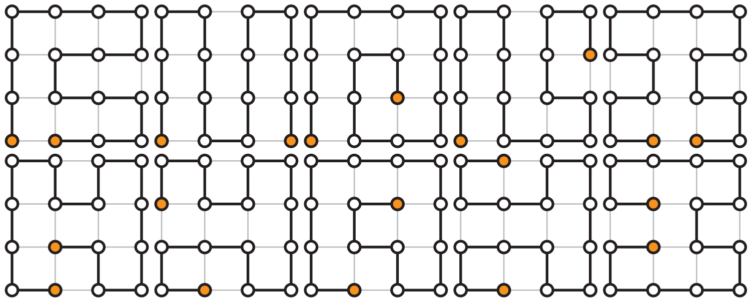

Figure 5. $G_{4,4}$ is traversable.

Lemma. All systems are traversable.

Proof. Since each system is isomorphic to $G_{4,4}$, it suffices to show that $G_{4,4}$ is traversable. Although there are 128 ways to choose starting and ending vertices with opposite colors, when we take symmetry into account, we need display only the 10 cases in figure 5.

Theorem. The knight's tour graph G is traversable.

Proof. Our construction considers three cases, depending on the locations of the original vertex o and the ending vertex e.

Since the start and end vertices have opposite colors, we may assume without loss of generality that o is a black vertex and e white. Call our starting system A_1, the other system of the same type A_2, and the remaining systems B_1 and B_2. For example, if A_1 is the d-system, then A_2 is the D-system. We will constructively prove the tour can be completed by considering three cases, depending on whether e is in A_1, A_2, or one of the B systems.

Case 1: Opposite system types

Suppose e is in one of the B systems, say B_2. For example, o is in a diamond system and e is in a square system. Using our lemma, we can traverse A_1 starting at o and ending on a white middle vertex, denoted m_1. We denote this by $o \xrightarrow{A_1} m_1$.

Next, go from m_1 to B_1, landing on a black vertex. Then traverse B_1, ending on a white middle vertex m_2. Move to A_2, and traverse it, ending on a white middle vertex m_3. Finally, jump to system B_2, making sure not to land on e. Since m_3 is adjacent to at least two vertices in B_2, this is always possible. We will enter B_2 on a black vertex and, by our lemma, can traverse B_2 ending on e, thus completing the tour. To summarize case 1, we have

$$o \xrightarrow{A_1} m_1 \xrightarrow{B_1} m_2 \xrightarrow{A_2} m_3 \xrightarrow{B_2} e.$$

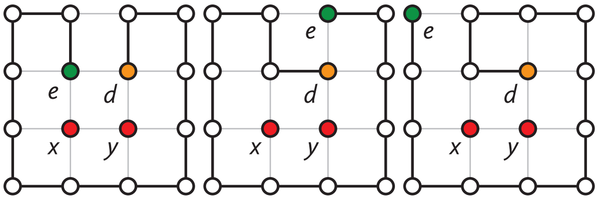

Figure 6. $G_{4,4}$ is semitraversable.

Case 2: Systems of the same type

For this case, we proceed similarly to the first case, with the exception that e is in A_2, not B_2. For example, o and e are in different diamond systems.

Lemma. Given any vertex e in $G_{4,4}$, we can find three middle vertices—d, x, and y—where x and y are adjacent, and a 13-step path from d to e that reaches every vertex except x and y. This path is called a *semitraversal* of $G_{4,4}$.

Proof. By symmetry, we need consider only three cases, depending on where e is on $G_{4,4}$. The end point e will be one of the four middle vertices, one of the eight edge vertices, or one of the four corner vertices. We may choose our starting point d and our exempt vertices x and y based on those three cases. The choices and paths are presented in figure 6. Note that d and e have opposite colors, as do x and y.

We can now prove case 2. Before beginning, note whether e is a middle, corner, or edge vertex in A_2, and choose middle vertices d, x, and y as in the previous lemma. Let T denote the semitraversal of A_2 that starts at d, ends at e, and avoids x and y. Since x and y have opposite colors, let x denote the black vertex.

Starting at o, traverse A_1 and B_1 as before, ending on white middle vertices m_1 and m_2, respectively, where m_2 is adjacent to x. Next, move to x, then y, and then to B_2. We enter B_2 on a black vertex.

Traverse B_2 as normal, ending on a white middle vertex m_3, where m_3 is adjacent to d. Finally, move from m_3 to d, then proceed with semitraversal T from d to e, which covers all of A_2, except the already-visited x and y. We summarize case 2 in the diagram

$$o \xrightarrow{A_1} m_1 \xrightarrow{B_1} m_2 \rightarrow x \rightarrow y \xrightarrow{B_2} m_3 \rightarrow d \xrightarrow{A_2-\{x,y\}} e.$$

Case 3: Same system

Assume o and e are both in A_1. First, assume e is not one of the white corner vertices (a8 or h1). Then there is at least one vertex outside of A_1 that can reach e in one move. Let this vertex be f, which is in B_2, say.

Begin traversing A_1 as if we were going to end at e, but stop at the penultimate vertex, n_1. Suppose n_1 is not a corner vertex. Since e can reach B_2, n_1 can reach a white vertex in B_1. We then traverse B_1, then A_2, then B_2, making sure to end on f. Then we finish our traversal by going from f to e. We summarize this traversal as follows:

$$o \xrightarrow{A_1-\{e\}} n_1 \xrightarrow{B_1} m_2 \xrightarrow{A_2} m_3 \xrightarrow{B_2} f \rightarrow e.$$

The procedure fails if n_1 is a corner vertex, since it cannot reach B_1. In this case, n_1 is adjacent only to e and some other vertex n_0. Suppose

$$o \rightarrow \ldots \rightarrow n \rightarrow n_0 \rightarrow n_1 \rightarrow e$$

is the A_1 traversal. Perform only the first 12 jumps, stopping on black vertex n. Let f_0 be a neighbor of n_0 in B_2. Then proceed as follows:

$$o \xrightarrow{A_1-\{n_0,n_1,e\}} n \xrightarrow{B_1} m_2 \xrightarrow{A_2} m_3 \xrightarrow{B_2} f_0 \rightarrow n_0 \rightarrow n_1 \rightarrow e.$$

Finally, consider the case where e is a corner vertex. Proceed in similar fashion. Let f_1 be a point in B_2 adjacent to n_1. Then our traversal is as follows:

$$o \xrightarrow{A_1-\{n_1,e\}} n_0 \xrightarrow{B_1} m_2 \xrightarrow{A_2} m_3 \xrightarrow{B_2} f_1 \rightarrow n_1 \rightarrow e.$$

Thus, we have covered all three cases, and therefore, the challenging knight's tour is always solvable! ■

Arthur Benjamin teaches math at Harvey Mudd College and is a former editor of Math Horizons. *This paper combines some of his favorite passions: games, puzzles, math, and magic.*

Sam K. Miller graduated with a math degree from Harvey Mudd College in 2017 and is now a platform engineer at Supplyframe. When he's not diving deep into the endless world of math, he can be found coding, playing jazz saxophone, canoeing, DJing, powerlifting, or playing chess.

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Solution to puzzle on page 2

1 S	2 I	3 N	4 E	■	5 O	6 S	7 L	8 O	■	9 A	10 X	11 I	12 O	13 M
14 A	T	A	N	■	15 R	O	I	L	■	16 L	O	C	K	E
17 N	O	T	I	■	18 A	U	L	D	■	19 C	O	O	R	S
20 G	O	L	D	21 E	N	R	A	T	22 I	O	■	23 N	A	H
■	■	■	■	24 L	G	E	■	25 I	S	A	26 Y	■	■	■
27 S	28 T	29 O	30 L	I	■	31 R	32 O	M	E	■	33 A	34 S	35 H	36 E
37 T	H	E	A	S	38 P	■	39 P	E	E	■	40 W	I	E	N
41 P	A	S	C	A	L	42 S	T	R	I	43 A	N	G	L	E
44 A	N	T	I	■	45 U	P	I	■	46 T	R	E	M	O	R
47 T	E	E	N	■	48 R	A	N	49 K	■	50 E	R	A	T	O
■	■	■	51 G	52 U	A	C	■	53 E	54 O	S	■	■	■	■
55 O	56 F	57 T	■	58 K	L	E	59 I	N	B	O	60 T	61 T	62 L	63 E
64 H	O	O	65 H	A	■	66 A	G	O	G	■	67 A	R	E	S
68 M	I	D	I	S	■	69 G	O	B	Y	■	70 P	O	E	T
71 S	L	O	P	E	■	72 E	R	I	N	■	73 S	U	R	D