

Counting on Determinants

Arthur T. Benjamin and Naomi T. Cameron

1. THE PROBLEM OF THE DETERMINED ANTS. Imagine four determined ants who simultaneously walk along the edges of the picnic table graph of Figure 1. The ants can move only to the right (northeast, southeast, and sometimes due east) with the goal of reaching four different morsels.

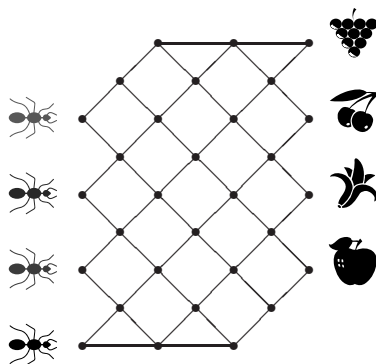


Figure 1. Ants and morsels.

Question 1. How many ways can the ants in Figure 1 simultaneously reach different morsels?

See Figure 3 for an example. We define an n -path to be a collection of n paths from a set of n origins to a set of n destinations. To compute the number of 4-paths in our example, we first find the number of ways that each ant can reach each morsel. In the example these numbers can be computed easily using calculations similar to those that arise in Pascal's triangle (see Figure 2). We record the information in a square matrix A whose (i, j) -entry a_{ij} is the number of ways that Ant i can reach Morsel j .

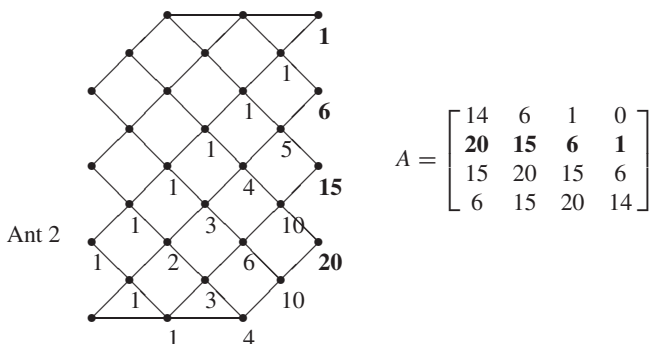


Figure 2. The number of paths for Ant 2 to reach each morsel can be computed recursively and is recorded in the second row of matrix A .

Consequently, the number of 4-paths such that each Ant i reaches Morsel i is $a_{11}a_{22}a_{33}a_{44} = 44100$. Alternatively, for each permutation π of $\{1, 2, 3, 4\}$ there are $a_{1\pi(1)}a_{2\pi(2)}a_{3\pi(3)}a_{4\pi(4)}$ 4-paths where each Ant i reaches Morsel $\pi(i)$. Summing over all alternatives gives us

$$\sum_{\pi \in S_4} a_{1\pi(1)}a_{2\pi(2)}a_{3\pi(3)}a_{4\pi(4)}$$

4-paths, where S_4 is the set of all permutations of $\{1, 2, 3, 4\}$. In other words, the “answer” to Question 1 is the *permanent* of the matrix A . In our example, this equals 171361.

Question 2. How many ways can the ants simultaneously reach different morsels, where no two paths intersect?

Notice that a feature of our picnic table graph is that in order to have four non-intersecting paths, we must have each Ant i go to Morsel i . Such a graph is called *nonpermutable*. However, most of the 44100 4-paths associated with the identity permutation do intersect somewhere. Believe it or not, the answer to Question 2 is the *determinant* of A , in this case 889.

To see why this is true, first recall that for an n -by- n matrix A the determinant of A equals

$$\sum_{\pi \in S_n} \text{sgn}(\pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)},$$

where S_n is the set of permutations of $\{1, 2, \dots, n\}$ and $\text{sgn}(\pi) = 1$ when π is an even permutation (expressible as the product of an even number of transpositions) and $\text{sgn}(\pi) = -1$ when π is odd. In terms of our counting problem, the determinant is a weighted sum, over all n -paths, where the weight of an n -path from C_1, \dots, C_n to respective destinations $D_{\pi(1)}, \dots, D_{\pi(n)}$ is the sign of π . In a nonpermutable graph, all of the nonintersecting n -paths are associated with the identity permutation, which is even, and thus given positive weight. It remains to prove that for every *intersecting* n -path, we can uniquely identify another intersecting n -path with opposite sign.

For a given intersecting n -path with associated permutation π , suppose i is the smallest index for which Path i intersects another path, and let Path j be the largest indexed path that Path i intersects. Let O be the first point of intersection of Paths i and j . For the 4-path in Figure 3, $i = 1$ and $j = 4$. To create an intersecting n -path

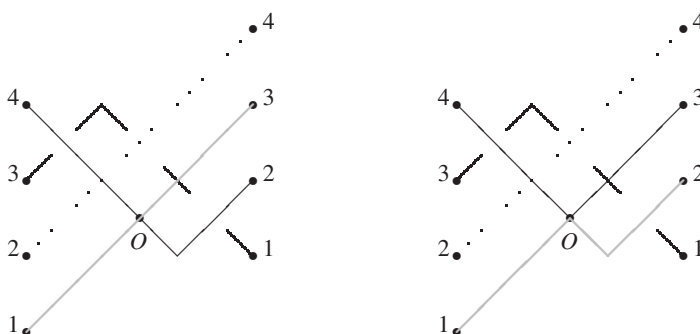


Figure 3. An intersecting 4-path with even permutation $\pi = (13)(24)$ is transformed into another intersecting 4-path with odd permutation $\pi' = (1243)$.

with opposite sign, we simply swap Paths i and j after O . Thus, Path i ends up at $D_{\pi(j)}$, Path j ends up at $D_{\pi(i)}$, and all other paths remain unchanged. Hence, the new n -path will have associated permutation $\pi' = (i, j)\pi$, which necessarily has opposite sign. In Figure 3, $\pi = (13)(24)$ is even and $\pi' = (14)(13)(24) = (1243)$ is odd. Notice that in the new n -path i , j , and O are the same as before, so $(\pi')' = \pi$, and the process is completely reversible.

The preceding argument applies to any directed graph that is *acyclic* (i.e., has no cycles) and leads to the following theorem:

Theorem 1. *Let G be a directed acyclic graph with n designated origin and destination nodes, and let A be the n -by- n matrix whose (i, j) -entry is the number of paths from the i th origin to the j th destination. The following statements hold:*

- (a) *The number of n -paths is equal to the permanent of A .*
- (b) *If G is nonpermutable, the number of nonintersecting n -paths is equal to the determinant of A .*
- (c) *In general, even if G is not nonpermutable, the determinant of A equals $\text{Even}(G) - \text{Odd}(G)$, where $\text{Even}(G)$ is the number of nonintersecting n -paths corresponding to even permutations and $\text{Odd}(G)$ is the number of nonintersecting n -paths corresponding to odd permutations.*

This theorem was originally given by Karlin and McGregor [5] and Lindstrom [7], and popularized by Gessel and Viennot [4], Aigner [1], and the recent book by Bressoud [2]. When the given graph G has considerable structure (as in cases where it is used to enumerate Young tableaux, plane partitions, or rhombus tilings), it is often possible to find a closed form for the determinant of A . Krattenthaler has described several methods for evaluating such determinants in [6].

Application. Recall that for $n \geq 0$ the n th Catalan number

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

counts the lattice paths from $(0, 0)$ to (n, n) restricted to vertices that stay on or below the line $y = x$. Let H be the $(n+1)$ -by- $(n+1)$ matrix with (i, j) -entry C_{i+j} ($0 \leq i, j \leq n$). In Figure 4, the number of paths from origin O_i to destination D_j is C_{i+j} . Since there is only one way to create $n+1$ nonintersecting paths

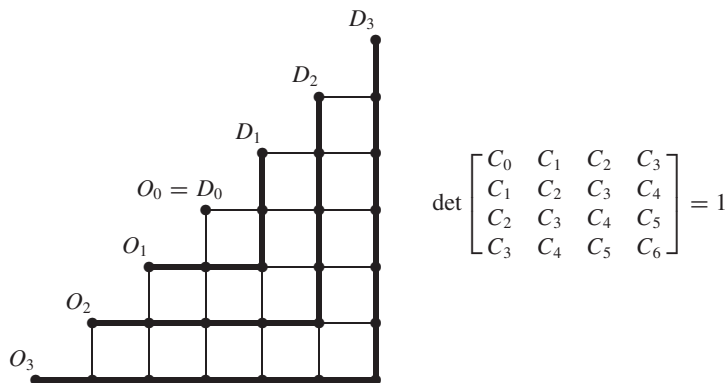


Figure 4. There is only one nonintersecting 4-path from $\{O_0, O_1, O_2, O_3\}$ to $\{D_0, D_1, D_2, D_3\}$.

from $\{O_0, O_1, \dots, O_n\}$ to $\{D_0, D_1, \dots, D_n\}$, Theorem 1 implies that $\det(H) = 1$. For another quick combinatorial derivation using matrix factorization, see [12].

2. ENUMERATING SPANNING TREES. Now consider the graph G in Figure 5. A *spanning tree* of G is a connected acyclic subgraph of G with the same vertex set.

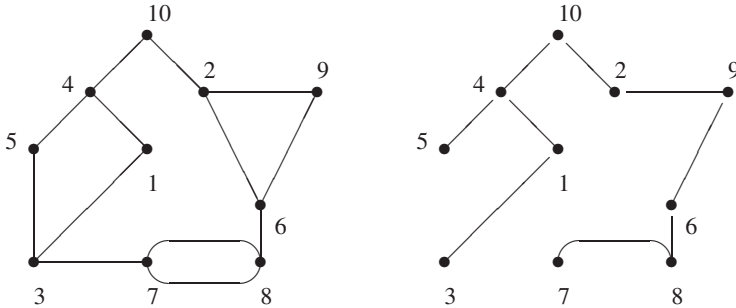


Figure 5. A graph G (left) and one of its many spanning trees (right).

Question 3. How many spanning trees does G have?

To answer this question, we first describe the graph in matrix notation. We allow our graph to contain multiple edges (e.g., between vertices 7 and 8 in G) but no *loops* (edges that begin and end at the same vertex). For a graph with n vertices, let A be its n -by- n adjacency matrix, where a_{ij} is the number of edges between i and j . Let D be the diagonal matrix $\text{Diag}(d_1, d_2, \dots, d_n)$, where d_i is the degree of vertex i (i.e., the number of edges incident at i). Now consider the matrix $D - A$. Since every row of $D - A$ sums to zero, its columns are linearly dependent; hence, $\det(D - A) = 0$. However, if we remove *any* row and column from $D - A$, the determinant of this submatrix answers our question. Specifically, we have:

Theorem 2. Let G be a loopless undirected graph with n vertices, adjacency matrix A , and diagonal degree matrix D . If $B_{r,s}$ signifies the $(n - 1)$ -by- $(n - 1)$ matrix obtained by deleting from $D - A$ its r th row and s th column, then the number of spanning trees of G is equal to $(-1)^{r+s} \det B_{r,s}$ for any choice of r and s .

This result is known as the “Matrix-Tree Theorem” and was originally proved by Sylvester [15] (see [17], [14] for algebraic proofs). We present a combinatorial proof (as given in [3]) of the case when $r = s = n$, whereupon the number of spanning trees is simply the determinant of B_{nn} , a matrix that we abbreviate as B .

For example, the graph in Figure 5 has

$$D - A = \begin{bmatrix} 2 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 \\ -1 & 0 & 3 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 3 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -2 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

The 9-by-9 matrix B obtained by removing the last row and column from $D - A$ is seen in Figure 10 (with its nonzero entries expanded). It has determinant 148. Hence the matrix-tree theorem tells us that G has 148 spanning trees.

Our proof of Theorem 2 is similar to the one in the previous section. Here, we will identify spanning trees as the acyclic objects in a large collection of directed graphs. The determinant will count all the acyclic objects and half of the cyclic objects *positively*, while the other half of the cyclic objects will be counted *negatively*. When the dust settles, only the spanning trees will remain standing.

For any spanning tree of G with n vertices, there is exactly one way to orient its edges so that each edge points in the direction of vertex n (see Figure 6). This is called a *rooted spanning tree* with root n . Hence, the number of spanning trees of G is equal to the number of rooted spanning trees of G with root n .

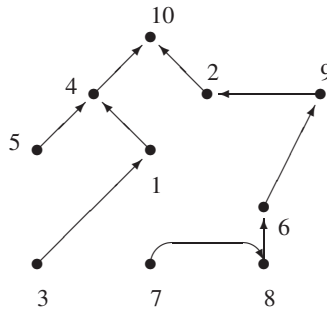


Figure 6. The spanning tree of Figure 5 rooted at vertex $n = 10$.

Observe that the rooted spanning trees of G have $n - 1$ edges, where each nonroot vertex i has *outdegree* 1. A directed subgraph of G with this property is called a *functional digraph* in G , since it represents a function $f : \{1, \dots, n - 1\} \rightarrow \{1, \dots, n\}$, where the edge from i to j indicates that $f(i) = j$. Let \mathcal{F} denote the set of all functional digraphs in G . Notice that the size of \mathcal{F} is $d_1 d_2 \cdots d_{n-1}$ (see Figure 7 for a typical example). From any vertex, we are ultimately led either to the root n or to a cycle that does not contain n . The acyclic objects of \mathcal{F} are precisely the rooted spanning trees of G .

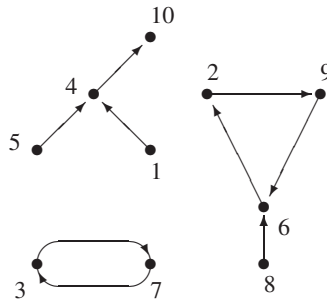


Figure 7. A typical functional digraph F in G .

Next, we consider an even larger collection \mathcal{S} of *signed* objects: \mathcal{S} comprises all functional digraphs F in G , where now each cycle in F is given a sign, either $+$ or $-$. Thus a functional digraph F in \mathcal{F} with k cycles generates 2^k *signed functional*

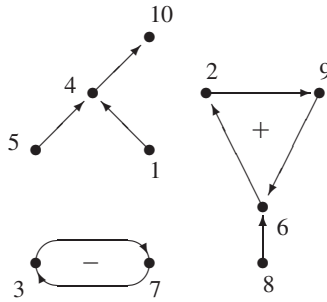


Figure 8. S is a signed functional digraph based on F from Figure 7. Here, the sign of S is negative.

digraphs in \mathcal{S} (see Figure 8). For S in \mathcal{S} we define the *sign* $\text{sgn}(S)$ of S to be the product of the signs of the cycles of S . If S has no cycles, then S is a rooted spanning tree and its sign is necessarily positive. If S is *cyclic* (that is, S has at least one cycle), then we define its *conjugate* \bar{S} to be the same functional digraph as S but with the sign of the first cycle reversed. We define the *first cycle* to be the cycle containing the vertex with the smallest label (see Figure 9). Notice that $\text{sgn}(\bar{S}) = -\text{sgn}(S)$ and $\bar{\bar{S}} = S$. This correspondence proves that there are as many positive cyclic elements as negative cyclic elements in \mathcal{S} . Hence the number of positive elements of \mathcal{S} minus the number of negative elements gives the number of acyclic elements of \mathcal{S} , which is the number of spanning trees of G .

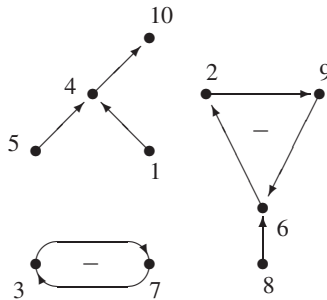


Figure 9. \bar{S} , the conjugate of S from Figure 8, has positive sign.

So how does the matrix B fit into this? Suppose that an edge between i and j exists in G , where $i \neq n$ and $j \neq n$. Then the directed edge from i to j is represented *twice* in B : positively, on the diagonal, as one of the 1s that comprise $b_{ii} = d_i = 1 + 1 + \dots + 1$, and negatively, off the diagonal, as one of the -1 s that comprise $b_{ij} = -1 - 1 \dots - 1$. (Note that $b_{ij} = -1$ unless there are multiple edges between vertices i and j .) A directed edge from i to n is represented only once, positively, as a 1 in b_{ii} . Thus, every signed functional digraph S , which consists of $n - 1$ directed edges, is associated with a selection of $n - 1$ 1s or -1 s in the matrix B , where each 1 or -1 comes from a different row and column of B . Specifically, for an edge e_{ij} in S directed from i to j that belongs to a negative cycle of S we select one of the -1 s in $b_{ij} = -1 - 1 \dots - 1$. Otherwise, we utilize the appropriate 1 in $b_{ii} = d_i = 1 + 1 + \dots + 1$ (see Figure 10 for an example).

Thus, when we expand each b_{ij} as the sum of 1s or -1 s, the determinant of B is represented as the sum of the products of a bunch of 1s, -1 s, and 0s. Each nonzero term corresponds to a signed functional digraph S in G . If S has exactly m edges in negative cycles, then its contribution to $\det(B)$ is $X_S = (-1)^m \text{sgn}(\pi_S)$, where π_S is

the permutation associated with S . Our goal is to show that $X_S = \text{sgn}(S)$, from which it follows that the determinant of B equals the number of positive elements of \mathcal{S} minus the number of negative elements of \mathcal{S} (i.e., the number of spanning trees of G).

$$B = \begin{bmatrix} \mathbf{1+1} & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1+1+1} & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & \mathbf{1+1+1} & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & \mathbf{1+1+1} & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & \mathbf{1+1} & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & \mathbf{1+1+1} & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & \mathbf{1+1+1} & -1-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1-1 & \mathbf{1+1+1} & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & \mathbf{1+1} \end{bmatrix}$$

Figure 10. The signed functional digraph S in Figure 8 corresponds to the selection of 9 bold 1s and -1 s in the matrix B .

To prove that $X_S = \text{sgn}(S)$, suppose that S contains k negative cycles C_1, \dots, C_k , where C_i contains m_i (≥ 2) edges. Thus, $\text{sgn}(S) = (-1)^k$, $m_1 + \dots + m_k = m$, and $\pi_S = \pi_1 \pi_2 \dots \pi_k$, where π_i is the cyclic permutation with m_i elements naturally described by C_i . For example, in Figure 8, $\pi_S = (37)$, whereas in Figure 9, $\pi_{\bar{S}} = (296)(37)$. Thus, since $\text{sgn}(\pi_i) = (-1)^{m_i-1}$, we have

$$\begin{aligned} X_S &= (-1)^m \text{sgn}(\pi_S) = (-1)^m \text{sgn}(\pi_1 \dots \pi_k) \\ &= (-1)^{m_1 + \dots + m_k} \text{sgn}(\pi_1) \dots \text{sgn}(\pi_k) \\ &= (-1)^{m_1 + \dots + m_k} (-1)^{m_1-1} \dots (-1)^{m_k-1} \\ &= (-1)^k = \text{sgn}(S), \end{aligned}$$

as was to be shown. ■

In the foregoing proof of Theorem 2, we assumed that $B_{r,s}$ ($= B_{nm}$) was obtained from $D - A$ by deleting its last row and column. If $r = s \neq n$, then the theorem can be proved in the same way by considering trees rooted at vertex s . If $r \neq s$, then a more sophisticated combinatorial proof is provided by Chaiken [3]. Other combinatorial arguments are also given by Orlin [11], Temperley [16], and Zeilberger [18].

The matrix-tree theorem can be extended in several directions, all of which can be proved by essentially the same combinatorial argument. If we let B^+ be the non-negative matrix $D + A$ with its last row and column removed, then the size of \mathcal{S} is the permanent of B^+ , since each signed functional digraph is counted once. Finally, notice what happens if we remove the last row and column of B , resulting in an $(n - 2)$ -by- $(n - 2)$ matrix C . An argument similar to the original one then reveals that the determinant of C counts the spanning *forests* (i.e., acyclic subgraphs) F of G with two connected components, where $n - 1$ and n are in different components of F . In general, we have:

Corollary 3. *Let G be a loopless, undirected graph with n vertices, adjacency matrix A , and diagonal degree matrix D . For a given set of vertices $V = \{v_1, \dots, v_k\}$ let B_V and B_V^+ denote the $(n - k)$ -by- $(n - k)$ matrices obtained by deleting rows and columns v_1, \dots, v_k from $D - A$ and $D + A$, respectively.*

- (a) The determinant of B_V counts the spanning forests F of G with k connected components, where the vertices v_1, \dots, v_k are in different components of F .
- (b) The permanent of B_V^+ counts the signed functional digraphs of G with exactly k connected components, each rooted at one of the vertices v_1, \dots, v_k .

Application. The complete graph K_n is a graph with n vertices and $\binom{n}{2}$ edges whose vertices are pairwise adjacent. Cayley's formula asserts that K_n has n^{n-2} spanning trees. To see this, observe that each vertex has degree $n - 1$, so $B = nI - J$ is the associated $(n - 1)$ -by- $(n - 1)$ matrix, where I is the identity matrix and J is the matrix of all ones. Since J has rank one, it has eigenvalue 0 with multiplicity $n - 2$ and eigenvalue $n - 1$ (with eigenvector $(1, 1, \dots, 1)$). Thus, $B = nI - J$ has eigenvalues n and 1, where n has multiplicity $n - 2$. Consequently, the number of spanning trees of K_n is $\det(B) = n^{n-2}$. The complete bipartite graph $K_{m,n}$ has $m + n$ vertices $v_1, \dots, v_m, w_1, \dots, w_n$ and mn edges, one edge for each v_i, w_j pair. As an exercise, we invite the reader to show that $K_{m,n}$ has $m^{n-1}n^{m-1}$ spanning trees [8].

3. PERMUTATIONS WITH SPECIFIED DESCENTS. In this section, we count restricted arrangements of numbers. The arrangement 382469157 has *descents* occurring in positions two and six, since the second number and sixth number are immediately followed by smaller numbers.

Question 4. How many arrangements of $\{1, 2, \dots, 9\}$ are possible, with the restriction that a descent is *allowed* at positions two, six, or seven but at no other positions?

For example, 382469157 is counted among the valid arrangements, as is 123456789, since it has no descents. The answer to Question 4 is the multinomial coefficient

$$\binom{9}{2} \binom{7}{4} \binom{3}{1} \binom{2}{2} = \frac{9!}{2!4!1!2!} = 3780,$$

since this counts the ways to select two numbers to occupy positions one and two, four numbers to occupy positions three through six, one number to occupy position seven, and two numbers to occupy the last two positions, with each selection of numbers written in ascending order. In general, if $S = \{s_1, \dots, s_k\}$ is the set of positions where a descent is allowed in an arrangement of $\{1, 2, \dots, n - 1\}$, then the number of valid arrangements is

$$f(n; S) = \frac{n!}{s_1!(s_2 - s_1)! \cdots (s_k - s_{k-1})!(n - s_k)!}. \quad (1)$$

Question 5. How many arrangements of the numbers $\{1, 2, \dots, 9\}$ are possible, with the restriction that a descent *must occur* at positions two, six, and seven but at no other positions?

We can easily compute the answer

$$\frac{9!}{2!4!1!2!} - \left[\frac{9!}{6!1!2!} + \frac{9!}{2!5!2!} + \frac{9!}{2!4!3!} \right] + \left[\frac{9!}{7!2!} + \frac{9!}{6!3!} + \frac{9!}{2!7!} \right] - \frac{9!}{9!} = 1667$$

using the principle of inclusion-exclusion as follows. From the set of $f(9; \{2, 6, 7\})$ arrangements previously considered, we subtract those arrangements without descents

in positions two, six, and seven, respectively; we add back arrangements without descents in positions two and six, two and seven, and six and seven, respectively; finally, we subtract the single arrangement without any descents. Each of these subproblems can be computed using equation (1). For example, the first subtracted term, which counts arrangements without descents in position two, but with possible descents in positions six and seven, is equal to $f(9; \{6, 7\}) = 9!/(6!1!2!)$.

In general, let $e(n; S)$ count the arrangements of the numbers 1 through n with descents occurring precisely at positions s_1, \dots, s_k . Then inclusion-exclusion gives us

$$e(n; S) = \sum_{T \subseteq S} (-1)^{k-|T|} f(n; T). \quad (2)$$

As we will soon see, $e(n; S)$ can be computed with the aid of a determinant. For example,

$$e(9; \{2, 6, 7\}) = 9! \det \begin{bmatrix} \frac{1}{2!} & \frac{1}{6!} & \frac{1}{7!} & \frac{1}{9!} \\ 1 & \frac{1}{4!} & \frac{1}{5!} & \frac{1}{7!} \\ 0 & 1 & \frac{1}{1!} & \frac{1}{3!} \\ 0 & 0 & 1 & \frac{1}{2!} \end{bmatrix}.$$

In general, we have:

Theorem 4. For a given subset $S = \{s_1, \dots, s_k\}$ of $\{1, \dots, n-1\}$ let C be the $(k+1)$ -by- $(k+1)$ matrix defined as follows: if $i > j+1$, $c_{ij} = 0$; if $i \leq j+1$, $c_{ij} = 1/(s_j - s_{i-1})!$, where $s_0 = 0$ and $s_{k+1} = n$. Then

$$e(n; S) = n! \det(C)$$

counts the arrangements of $\{1, \dots, n\}$ with descents occurring precisely at positions s_1, \dots, s_k .

The matrix C is almost upper-triangular with $c_{j+1,j} = 1$ ($1 \leq j \leq k$) on its subdiagonal and only 0s below the subdiagonal. Here, we have

$$n! \det(C) = \sum_{\pi \in \Pi} n! \operatorname{sgn}(\pi) c_{1\pi(1)} c_{2\pi(2)} \cdots c_{k+1\pi(k+1)}, \quad (3)$$

where Π denotes the set of all permutations of $\{1, \dots, k+1\}$ that satisfy $\pi(j+1) \geq j$ for $j = 1, \dots, k$. All other permutations result in a product of 0.

To prove Theorem 4, we prove that the summands in equations (2) and (3) are identical. Notice that (2) involves 2^k summands. We claim that equation (3) also has 2^k summands by finding a one-to-one correspondence between subsets of $\{1, \dots, k\}$ and Π . Indeed, if J is a subset of $\{1, \dots, k\}$ with complement J^c , then we associate with J the permutation π in Π that satisfies $\pi(j+1) > j$ if and only if j belongs to J . In other words, π is completely determined by J^c , the columns from which we select subdiagonal 1s.

For example, suppose that $k = 20$ and $J = \{4, 10, 11, 18\}$. We construct the unique permutation π that chooses the subdiagonal element $c_{j+1,j} = 1$ for every column j (i.e., $\pi(j+1) = j$), *except* when j is a member of $\{4, 10, 11, 18, 21\}$. Since 1, 2, and 3 are not elements of J , we must choose the subdiagonal 1s in columns 1, 2, and 3, which are the subdiagonal entries of rows 2, 3, and 4. Thus, from column 4, which

corresponds to a member of J , we may not choose the entries of rows 2, 3, 4, or 5, for c_{54} is on the subdiagonal. Hence c_{14} is the only eligible nonzero term in column 4. Thus, we must have

$$\pi(1) = 4, \quad \pi(2) = 1, \quad \pi(3) = 2, \quad \pi(4) = 3. \tag{4}$$

Since 10 is the next element of J , we are forced to choose subdiagonal 1s in columns 5, 6, 7, 8, and 9 or, equivalently, in rows 6, 7, 8, 9, and 10. Thus, we cannot select the entry $c_{i,10}$ for $i = 6, 7, 8, 9$, or 10, nor for $i = 1, 2, 3$, or 4 by (4), nor for $i = 11$, since 10 belongs to J . Accordingly, $c_{5,10}$ is the only eligible choice in column 10. We infer that

$$\pi(5) = 10, \quad \pi(6) = 5, \quad \pi(7) = 6, \quad \pi(8) = 7, \quad \pi(9) = 8, \quad \pi(10) = 9.$$

Continuing with this logic, we must also select $c_{11,11}, c_{12,18}, c_{18,21}$; all other selections are subdiagonal 1s. Consequently, in this example, the unique product associated with the columns of J is

$$\begin{aligned} n! c_{14}c_{5,10}c_{11,11}c_{12,18}c_{19,21} &= \frac{n!}{s_4! (s_{10} - s_4)! (s_{11} - s_{10})! (s_{18} - s_{11})! (n - s_{18})!} \\ &= f(n; \{4, 10, 11, 18\}). \end{aligned}$$

Here, the associated permutation is

$$\pi = (1, 4, 3, 2)(5, 10, 9, 8, 7, 6)(11)(12, 18, 17, 16, 15, 14, 13)(19, 21, 20),$$

which has sign $(-1)^{21-5} = 1$.

In general, with a subset $J = \{j_1, \dots, j_m\}$ of $\{1, \dots, k\}$ we associate π_J in Π , where $\pi_J(j+1) = j$ for j in J^c , and

$$\begin{aligned} \pi_J(1) &= j_1, \\ \pi_J(j_1 + 1) &= j_2, \dots, \pi_J(j_{m-1} + 1) = j_m, \\ \pi_J(j_m + 1) &= k + 1. \end{aligned}$$

Since π_J has $k + 1$ elements in $m + 1$ cycles, $\text{sgn}(\pi_J) = (-1)^{k-m} = (-1)^{k-|J|}$. Letting $T_J = \{s_{j_1}, \dots, s_{j_m}\}$, it follows that

$$\begin{aligned} n! \det(C) &= \sum_{\pi_J \in \Pi} \text{sgn}(\pi_J) n! c_{1,j_1} c_{j_1+1,j_2} \cdots c_{j_{m-1}+1,j_m} c_{j_m+1,k+1} \\ &= \sum_{\pi_J \in \Pi} (-1)^{k-|J|} \frac{n!}{s_{j_1}! (s_{j_2} - s_{j_1})! \cdots (s_{j_m} - s_{j_{m-1}})! (n - s_{j_m})!} \\ &= \sum_{T_J \subseteq S} (-1)^{k-|T_J|} f(n; T_J), \end{aligned}$$

as desired. ■

This theorem was originally proved by MacMahon [9]. It is interesting to note that Theorem 4 can also be proved as an application of Theorem 1, as demonstrated by

Gessel and Viennot [4]. An extension to determinants with polynomial entries (q -binomial coefficients) is given by Stanley [13].

Application. There is only one permutation of $\{1, \dots, n\}$ with a descent at every position but the last one. That is, $e(n; \{1, 2, \dots, n-1\}) = 1$. For the n -by- n matrix C with $c_{ij} = 1/(j-i+1)!$ if $i \leq j+1$ and $c_{ij} = 0$ otherwise we conclude that $n! \det(C) = 1$. For example,

$$\det \begin{bmatrix} 1 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} \\ 1 & 1 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} \\ 0 & 1 & 1 & \frac{1}{2!} & \frac{1}{3!} \\ 0 & 0 & 1 & 1 & \frac{1}{2!} \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \frac{1}{5!}.$$

More combinatorial approaches to linear algebra are presented by Zeilberger [18].

ACKNOWLEDGMENTS. The authors thank Ira Gessel and Louis Shapiro for many helpful remarks.

REFERENCES

1. M. Aigner, Lattice paths and determinants, *Computational Discrete Mathematics*, 1–12, Lecture Notes in Comput. Sci., no. 2122, Springer-Verlag, Berlin, 2001.
2. D. M. Bressoud, *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture*, Mathematical Association of America, Washington, D.C., 1999.
3. S. Chaiken, A combinatorial proof of the all-minors matrix tree theorem, *SIAM J. Algebraic Discrete Methods* **3** (1982) 319–329.
4. I. Gessel and G. Viennot, Binomial determinants, paths, and hook length formulae, *Adv. in Math.* **58** (1985) 300–321; see also Determinants, paths, and plane partitions (preprint, 1989) by these authors.
5. S. Karlin and G. McGregor, Coincidence probabilities, *Pacific J. Math.* **9** (1959) 1141–1164.
6. C. Krattenthaler, Advanced determinant calculus, article B42q in *Proceedings of the Seminaire Lotharingien de Combinatoire*, issue 42 (The Andrews Festschrift), 1998, 67 pp.; available at <<http://www.mat.univie.ac.at/~slc/>>.
7. B. Lindstrom, On the vector representations of induced matroids, *Bull. London Math. Soc.* **5** (1973) 85–90.
8. L. Lovasz, *Combinatorial Problems and Exercises*, 2nd ed., North-Holland, Amsterdam, 1993.
9. P. A. MacMahon, Second memoir on the compositions of numbers, *Philos. Trans. Roy. Soc. London Ser. A* **207** (1908) 65–134.
10. J. W. Moon, Some determinant expansions and the matrix-tree theorem, *Discrete Math.* **124** (1994) 163–171.
11. J. B. Orlin, Line-graphs, arborescences, and theorems of Tutte and Knuth, *J. Combin. Theory Ser. B* **25** (1978) 187–198.
12. L. W. Shapiro, A Catalan triangle, *Discrete Math.* **14** (1976) 83–90.
13. R. P. Stanley, *Enumerative Combinatorics*, vol. 1, Cambridge University Press, Cambridge, 1997.
14. _____, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, Cambridge, 1999.
15. J. J. Sylvester, On the change of systems of independent variables, *Quart. J. Math.* **1** (1857) 42–56; also in *Collected Mathematical Papers*, vol. 2, Cambridge University Press, Cambridge, 1908, pp. 65–85.
16. H. N. V. Temperley, *Graph Theory and Its Applications*, Ellis Horwood Series in Mathematics and its Applications, Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons], New York, 1981.
17. D. B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, NJ, 2001.
18. D. Zeilberger, A combinatorial approach to matrix algebra, *Discrete Math.* **56** (1985) 61–72.

ARTHUR T. BENJAMIN earned his B.S. in applied mathematics from Carnegie Mellon University in 1983 and his Ph.D. in mathematical sciences from Johns Hopkins University in 1989. Since then, he has taught at Harvey Mudd College, where he is a professor of mathematics. In 2000, he was awarded the MAA's Haimo

Award for Distinguished College Teaching. In 2003, he coauthored (with Jennifer Quinn) *Proofs That Really Count: The Art of Combinatorial Proof*, which is published by the MAA. He and Jennifer are coeditors of *Math Horizons* magazine.
Harvey Mudd College, Department of Mathematics, Claremont, CA 91711
benjamin@hmc.edu

NAIOMI T. CAMERON earned her B.S. and Ph.D. degrees from Howard University in 1995 and 2002, respectively. Her research interests are in discrete mathematics and combinatorics. She is currently an assistant professor of mathematics at Occidental College in Los Angeles.
Occidental College, Department of Mathematics, Los Angeles, CA 90041
ncameron@oxy.edu

Goldbach's Conjecture Implies Bertrand's Postulate

In recent years, there has been a great deal of renewed interest in *Goldbach's Conjecture* (GC) that every even integer greater than two is the sum of two primes, due primarily to the publication of a novel [1] and the substantial prize offered by its publishers for a proof. (The prize was not claimed by the deadline, and right now the best result is J. R. Chen's 1966 theorem that every sufficiently large integer is the sum of a prime and the product of at most two primes.)

An examination of several number theory texts, old and new, reveals that an interesting consequence of GC seems to have been overlooked: GC provides a short proof of *Bertrand's Postulate*, which states that for every positive integer greater than 1 there exists a prime p such that $n < p < 2n$. The proof is simple. Since $n > 1$, $2n$ is an even integer greater than two, whence $2n = p_1 + p_2$ for primes p_1 and p_2 . If both p_1 and p_2 are less than n , then $p_1 + p_2 < 2n$. Therefore, at least one of the p_i is greater than or equal to n —that is, $n \leq p_i < 2n$. If n is not a prime, strict inequality holds. On the other hand, if n is prime, then $n + 1$ is composite and $2(n + 1) = p'_1 + p'_2$, where p'_1 or p'_2 (say, p'_2) must be greater than $n + 1$ —that is, greater than n . Accordingly, $n < p'_2 < 2n + 2$. But p'_2 can't be $2n + 1$ (or else $p'_1 = 1$) nor can it be $2n$ (since $2n$ is composite), so $n < p'_2 < 2n$.

The usual proof (see, for example [2, pp. 367–368]) requires some reasonably sophisticated analytic estimates and is omitted from most elementary texts.

REFERENCES

1. A. Doxiadis, *Uncle Petros and Goldbach's Conjecture*, Bloomsbury, New York, 2000.
2. I. Niven, H. S. Zuckerman, and H. L. Montgomery, *An Introduction to the Theory of Numbers*, 5th ed., John Wiley & Sons, New York, 1991.

—Submitted by Henry J. Ricardo, Medgar Evers College (CUNY)
A similar item, involving essentially the same argument, was submitted by Yoshihiro Tanaka, Hokkaido University, Japan