The College Mathematics Journal

ISSN: 0746-8342 (Print) 1931-1346 (Online) Journal homepage: http://www.tandfonline.com/loi/ucmj20

Factoring Numbers with Conway's 150 Method

Arthur T. Benjamin

To cite this article: Arthur T. Benjamin (2018) Factoring Numbers with Conway's 150 Method, The College Mathematics Journal, 49:2, 122-125, DOI: 10.1080/07468342.2017.1411141

To link to this article: https://doi.org/10.1080/07468342.2017.1411141

Published online: 20 Feb 2018.

Factoring Numbers with Conway's 150 Method

Arthur T. Benjamin

Art Benjamin (benjamin@hmc.edu, MR ID 34635) received his Ph.D. from Johns Hopkins University and is currently the Smallwood Family Professor of Mathematics at Harvey Mudd College, where he has been teaching since 1989. His passions include backgammon, magic, racing calculators, writing song parodies, and spending time with his family.

In 1993, Thomas Rodgers hosted an event that celebrated Martin Gardner's achievements, called "Gathering For Gardner." I attended the second such gathering (G4G2) in 1996, where I had the pleasure of meeting Martin in person for the first time. The event attracted mathematically inclined magicians and magically inclined mathematicians, such as Persi Diaconis and John Horton Conway, along with an impressive number of puzzle people, philosophers, and artists from around the world. At this event, Conway shared with me his "150 method" for mentally finding prime factors of numbers that are not too large.

Mental math divisibility tests

It is easy to determine if a number is divisible by 2, 3, or 5. What about the next three primes 7, 11, and 13? If you are given a four, five, or six digit number, you can reduce the problem by testing the number obtained by taking the last three digits of the number and subtracting the number to its left. For example, to test 52,234, we test instead $234 - 52 = 182$. Since 182 is divisible 7 and 13, but not 11, the same is true for the original number. For the number 112,358, we compute $358 - 112 = 246$, which is not divisible by 7, 11, or 13, and so the same holds for the original number. Even with a number like 314,159, the difference between the first three digits and the last three digits, $314 - 159 = 155$, is not divisible by 7, 11, or 13, so neither is the original number. (Subtraction tip: To subtract 159, first subtract 200, then add back the complement, 41, i.e., $314 - 159 = 314 - 200 + 41 = 114 + 41 = 155$.)

The secret behind this magical method is that $7 \cdot 11 \cdot 13 = 1001$, so adding or subtracting a multiple of 1001 will not change a number's divisibility by any of those numbers. In the example above, to test divisibility of 52,234 by 7, 11, or 13, we can subtract $52 \cdot 1001 = 52{,}052$ to reach 182. Likewise, subtracting 314,314 from 314,159 allows us to test the number -155 instead.

By the way, there is a well-known method for determining divisibility by 11, for any sized number. You simply alternately subtract and add the digits of the original

doi.org/10.1080/07468342.2017.1411141
MSC: 11A51

number to produce a smaller number, and see if the smaller number is a multiple of 11. For example, since $3 - 1 + 4 - 1 + 5 - 9 = 1$ is not a multiple of 11, neither is 314,159. However, since $3 - 2 + 4 - 1 + 5 - 9 = 0$ is a multiple of 11, so is 324,159. The basis for this method is that, since we work in base 10 and $10 \equiv -1 \pmod{11}$, the $(n+1)$-digit number

$$a_n a_{n-1} \cdots a_1 a_0 = \sum_{k=0}^{n} a_k 10^k \equiv \sum_{k=0}^{n} a_k (-1)^k \pmod{11}.$$

There are simple “rip-off” tests for dividing by 7 and 13. To test divisibility by 7, you can rip off the last digit, double it, and subtract from the remaining number. For example, to test 3829, we reduce that number to $382 - 18 = 364$, which is reduced to $36 - 8 = 28$, which is a multiple of 7, and so the original number is too. This works because if $10a + b \equiv 0 \pmod 7$, then multiplying by 5 gives $50a + 5b \equiv 0 \pmod 7$, or equivalently, $a - 2b \equiv 0 \pmod 7$. Conversely, if we multiply the last congruence by 10, we arrive at the original congruence.

Similarly, one can show, by multiplying by 4, that $10a + b \equiv 0 \pmod{13}$ is equivalent to $a + 4b \equiv 0 \pmod{13}$, allowing us to test divisibility by 13 by ripping off the last digit, quadrupling it, and adding it to the remaining number. For example, the number 31,415 becomes $3141 + 20 = 3161$, which becomes $316 + 4 = 320$, and that becomes $32 + 0 = 32$, which is not a multiple of 13, so neither is 31,415.

One can likewise derive similar rip-off methods for other primes p that are especially handy when the number 10 has a reasonably small positive or negative multiplicative inverse modulo p, but that requires memorization. In practice, I use a method that works nicely for all divisors that are relatively prime to 10 (such as all primes bigger than 5). Here, we simply add or subtract a multiple of the divisor to the original number so that the sum or difference ends in 0, and then we delete the 0 to reach a smaller number. For example, to test if 12,345 is a multiple of 17, I might begin by adding $17 \cdot 5 = 85$ to reach 12,430, and then delete the 0 to get 1243. Then $1243 + 17 = 1260$, followed by $126 + 34 = 160$. Since 16 is not a multiple of 17, neither is 12,345.

Conway’s 150 method

The advantage of Conway’s 150 method is that it tests for many prime factors at the same time without the need to restart with the original number as we change trial divisors. It works especially well for testing 3- and 4-digit numbers and is well suited for hand calculations (on paper) or mental calculations (on your hand) as we explain. The method is based on the fact that the numbers 150 to 156 contain an unusual number of small primes as divisors. Using it, one can quickly determine if a 3- or 4-digit number is divisible by any prime number below 37. As we will show, applying this algorithm with some adjustments will allow you to find *all* prime factors of any number below $41^2 = 1681$. The key prime factorizations are given below, with the large prime factors highlighted, along with a “handy” mnemonic.

$152 = 2^3 \cdot \mathbf{19}$	(thumb)
$153 = 3^2 \cdot \mathbf{17}$	(index finger)
$154 = 2 \cdot \mathbf{7} \cdot \mathbf{11}$	(middle finger)
$155 = 5 \cdot \mathbf{31}$	(ring finger)
$156 = 2^2 \cdot 3 \cdot \mathbf{13}$	(pinky)

We note that the list contains all the primes below 41, except for 23, 29, and 37, but we will handle those later. To factor the number N, suppose the largest multiple of 150 below N is $150K$. Let $A = N - 150K$. Clearly, N is a multiple of 2, 3, or 5 if and only if A is a multiple of 2, 3, or 5, respectively. Now we count down by K's, while checking for the relevant prime factors along the way. Specifically, we check to see if $A - K$ is a multiple of 151 (we actually never need to perform that step), if $A - 2K$ is a multiple of 19, if $A - 3K$ is multiple of 17, if $A - 4K$ is a multiple of 7 or 11, if $A - 5K$ is a multiple of 31, and if $A - 6K$ is a multiple of 13. For example, if $N = 931$, then $A = 931 - 150 \cdot 6 = 31$, so N is not divisible by 2, 3, or 5. Next, we count down by 6, namely 31, 25, 19, 13, 7, 1, −5, obtaining "hits" at 19 and 7, which must therefore be the factors of 931. When counting down, Conway uses his hand to keep track of the relevant primes, so the thumb is associated with 19 (the big factor of 152), the index finger is associated with 17, the middle finger is 7 and 11, the ring finger is 31, and the pinky is 13. Thus, in the last example, you would touch your thumb when you mentally say 19, then the index finger with 13, the middle finger with 7, the ring finger with 1, and the pinky with −5.

Here is another example, along with a justification for why the method works: If $N = 689$, then $A = 689 - 150 \cdot 4 = 89$. Counting down by 4 gives 89, 85, 81 (thumb), 77 (index), 73 (middle), 69 (ring), and 65 (pinky). Since 65 is a multiple of 13, so is 689. The reason this test works is that $65 = N - 150K - 6K = N - 156K$ is a multiple of 13 if and only if N is a multiple of 13. Naturally, if you find a factor early on, it may pay to divide your number N by that factor to give yourself an easier problem. For example, since multiples of 2, 3, and 5 are obvious, one should probably divide by these numbers first before testing for higher factors. For example, with a number like 6,738, you should first divide the number by 6 to reduce the problem to determining the factors of 1123.

The same strategy applies if we round *up* to a multiple of 150, but this time we count *up* by K. For example, if $N = 1123$, then we can round up to $1200 = 150 \cdot 8$, so $A = 1200 - 1123 = 77$ and we count up by 8's: 77, 85, 93, 101, 109, 117, 125. Obtaining no hits, we conclude that 1123 is not divisible by 2, 3, 5, 7, 11, 13, 17, 19, or 31.

Modifications

Should you want to test for all primes below 37, Conway offers a modification to his procedure that will include divisibility checks for 23 and 29. Here, you round down to the nearest multiple of 300. Let $A = N - 150(2K)$. You will still count down by multiples of $2K$, but before doing so, take a "half-step" running start by testing the number $A + K$. Since $A + K = N - 299K$ and $299 = 13 \cdot 23$, this gives us a quick test for 13 and 23 (which he associates with the palm of his hand). Then, count down by $2K$ as before, $A - 2K, A - 4K, A - 6K, A - 8K, A - 10K$ (which equal $N - 302K$, $N - 304K, N - 306K, N - 308K, N - 310K$, respectively). But it is no longer necessary to subtract another $2K$ to reach $A - 12K = N - 312K$ since the factor of 13 has already been established at the beginning. Instead, you now add $10(2K)$ to obtain the number $N - 290K$. Hence, in the modified procedure, the pinky will be associated with 29. For example, if $N = 1007$, then $A = 1007 - 150(6) = 107$, so we will be counting down by 6. But first we test $107 + 3 = 110$ (for factors of 13 and 23), then we count down 101 (ignore), 95 (test 19), 89 (test 17) 83 (test 7, 11), and 77 (test 31), and then we jump up to $77 + 60 = 137$ (test 29). The only hit is 19, so the only prime factor of 1007 below 37 is 19. Indeed, $1007 = 19 \cdot 53$.

Essentially, the same procedure can be used when we round *up* to an even multiple of 150, except here you start by testing $A - K$, then count up from A by multiples of $2K$, with a subtraction of $20K$ in the last step. For example, to factor 899, we round up to 900, so that $A = 150(6) - 899 = 1$ and $2K = 6$. Thus, we test $A - K = -2$ for 13 and 23, then count 1, 7, 13 (test 19), 19 (test 17), 25 (tests 7 and 11), and 31 (test 31), then $31 - 60 = -29$ (test 29), obtaining hits on 31 and 29. Indeed, $899 = 29 \cdot 31$.

Finally, there is a quick test for multiples of 37, similar to the 7–11–13 method earlier, exploiting the fact that 37 is a divisor of 999. If the number has four, five, or six digits, then add the 3-digit number at the end of the number to the 1-, 2-, or 3-digit number at the beginning. For example, 62,123 is a multiple of 37 since 62 + 123 = 185 is a multiple of 37. (The reason this works is that we are essentially subtracting $62 \cdot 999$, which can be obtained by subtracting 62,000 and adding back 62.) Likewise, 234,432 is a multiple of 37 since $234 + 432 = 666$ is a multiple of 37. There are more tests of specific divisors using tests like the 37 test [1].

To test for prime factors up to 67, Conway uses his "2000 method," which exploits the fact that all primes up through 67 are represented between 1998 and 2021 (with more than half of them between 1998 and 2002). Specifically, listing the numbers in the order used by Conway,

$$\begin{aligned}
2000 &= 2^4 \cdot 5^3, \\
2001 &= 23 \cdot 29 \cdot 3, \\
2002 &= 7 \cdot 11 \cdot 13 \cdot 2, \\
1998 &= 37 \cdot 54, \\
2006 &= 17 \cdot 59 \cdot 2, \\
2010 &= 67 \cdot 30, \\
2014 &= 19 \cdot 53 \cdot 2, \\
2013 &= 61 \cdot 33, \\
2015 &= 31 \cdot 65, \\
2009 &= 41 \cdot 49, \\
2021 &= 43 \cdot 47.
\end{aligned}$$

Starting $A = N - 2000K$, we mostly count down by steps of K or $4K$ with the occasional up sidetrip, ending with an upstep and downstep of size $6K$. The 2000 method will factor all numbers below $71^2 = 5041$, but requires you to be comfortable at recognizing factors of most 3-digit numbers, whereas the 150 method only requires you to be able to factor most numbers below 200.

Acknowledgment. The author thanks the referee for many helpful suggestions, and especially wishes to thank John Conway for his permission to share his algorithm and for being such a large prime factor in the mathematics community.

Summary. We describe a "handy" method, due to John Conway, for quickly finding all relatively small prime factors of 3-digit and 4-digit numbers.

References

[1] Renault, M. (2006). Stupid divisibility tricks: 101 ways to stupefy your friends. *Math. Horiz.* 14(2): 18–21.