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FIBONACCI AND LUCAS IDENTITIES THROUGH COLORED TILINGS

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ABSTRACT. In this paper we introduce the concept of colored Fibonacci tilings which leads to charming combinatorial proofs of Fibonacci and Lucas number identites.

1. INTRODUCTION

Combinatorial proofs can lead to a greater appreciation and understanding for any topic. Fibonacci and Lucas identities are no exception. Let $F_n = F_{n-1} + F_{n-2}$, where $F_0 = 0$, $F_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$, where $L_0 = 2$, $L_1 = 1$. The identities

$$(1) \qquad \binom{n}{1} + 5\binom{n}{3} + 5^2\binom{n}{5} + \cdots + 5^k\binom{n}{2k+1} + \cdots = 2^{n-1}F_n$$

$$(2) \qquad \binom{n}{0} + 5\binom{n}{2} + 5^2\binom{n}{4} + \cdots + 5^k\binom{n}{2k} + \cdots = 2^{n-1}L_n$$

are usually proved by applying the binomial theorem to Binet's formulas $F_n = \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)/\sqrt{5}$ and $L_n = \left(\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$ [5]. While reading these identities, one cannot escape the feeling that the left hand side is counting *something*. In this paper, we tile boards and bracelets with colored squares and dominoes to achieve the desired combinatorial interpretation.

Recall that F_{n+1} counts the number of sequences of 1's and 2's which sum to n. Equivalently, F_{n+1} counts the number of ways to tile a $1 \times n$ rectangle (called an *n-board* consisting of *cells* labelled $1, ..., n$) with 1×1 *squares* and 1×2 *dominoes*. For combinatorial convenience, we define $f_n = F_{n+1}$. Then f_n is the number of *n-tilings*, the number of ways to tile an n-board with squares and dominoes.

Similarly, it is easy to show that L_n counts the number of ways to tile a *circular* n-board with squares and dominoes. Cells are labeled 1 though n and a tiling is called an *n-bracelet*. An n-bracelet is *out of phase* if the same domino covers cells n and 1; otherwise the n-bracelet is *in phase*. There are f_n in phase n-bracelets and f_{n-2} out of phase n-bracelets. Thus for $n \geq 2$, $L_n = f_n + f_{n-2}$.

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2. Colored Tilings

The motivation for this section comes from investigating the quantity $2^n f_n$. Given an n-board, there are 2^n to ways to paint each cell black or white. There are f_n ways to tile the board with"transparent" dominoes and squares. Looking through the tiles, there are 6 possible tile colorings, namely a white square, a black square, a white-white domino, a white-black domino, a black-white domino, and a black-black domino. Thus there are $2^n f_n$ ways to tile an n-board with two types of squares (black and white) and four types of dominoes (say red, yellow, green, and violet). We call this a *colored tiling*. Analogously, there are $2^n L_n$ colored n-bracelets.

Using colored tilings we create a two-to-one mapping to illustrate the identity

$$2^{n+1} f_n = \sum_{k=0}^{n} 2^k L_k.$$

We think of L_0 as representing two "empty" bracelets, one being in phase and the other out of phase. For every colored tiling of an n-board we generate two colored bracelets of length less than or equal to n. Let T be a colored tiling of an n-board. If T is not the all-white tiling, let k denote the last cell covered by a non-white tile. As Figure 1 illustrates, we generate two k-bracelets as follows:

B_1 : Remove cells $k+1$ through n. B_1 is the in phase k-bracelet ending with a non-white tile produced by gluing cells k and 1 together.

B_2 : *If cell k is covered by a black square*, then B_2 is the in phase k-bracelet obtained by replacing the kth cell of B_1 with a white square.
If cell k is covered by a colored domino, then B_2 is the out of phase k-bracelet obtained by rotating the tiles of B_1 clockwise one cell.

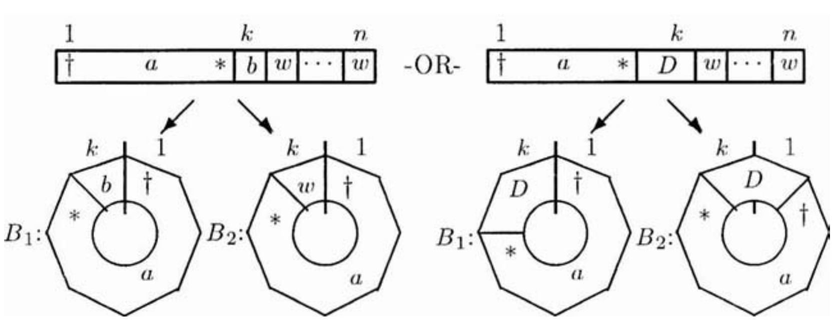

FIGURE 1. To prove $2^{n+1} f_n = \sum_{k=0}^{n} 2^k L_k$, we draw a two-to-one correspondence depending on the last cell covered by a non-white tile. w represents a white square, b represents a black square, and D represents a colored domino.

Every colored k-bracelet, where $1 \leq k \leq n$ is obtained exactly once in this fashion. The process is easily reversed by examining the last tile. The case where T consists of all white squares is identified with the two empty bracelets. Thus $2 \cdot 2^n f_n = \sum_{k=0}^{n} 2^k L_k$.

To tackle identity (1), we first replace n with $n+1$ to give us

$$\binom{n+1}{1} + 5\binom{n+1}{3} + 5^2\binom{n+1}{5} + \cdots + 5^k\binom{n+1}{2k+1} + \cdots = 2^n f_n.$$

By Pascal's identity, this becomes

$$\binom{n}{0} + \binom{n}{1} + 5\binom{n}{2} + 5\binom{n}{3} + 5^2\binom{n}{4} + \cdots + 5^{\lfloor \frac{t}{2} \rfloor}\binom{n}{t} + \cdots = 2^n f_n.$$

As before, $2^n f_n$ counts the number of colored tilings of an n-board. To show that this equals the left hand side, we provide coloring rules defined in such a way that for every subset $\{x_1, x_2, \ldots, x_t\}$ of $\{1, 2, \ldots, n\}$, we generate exactly $5^{\lfloor t/2 \rfloor}$ colored tilings of an n-board.

Assume $x_1 < x_2 < \cdots < x_t$, and that t is even. This gives rise to $t/2$ disjoint intervals $I_1 = [x_1, x_2]$, $I_2 = [x_3, x_4], \ldots, I_{t/2} = [x_{t-1}, x_t]$. Any cell not belonging to one of these intervals is colored by a white square. Inside an interval, we have 5 tiling choices. We may cover the interval entirely with squares where the end points must be black and the interior (if it exists) must be white. Otherwise we may cover the interval entirely with dominoes of the same color (when the interval has an even number of cells) or we may cover the interval with dominoes of the same color followed by a single black square (when the interval has an odd number of cells). See Figure 2.

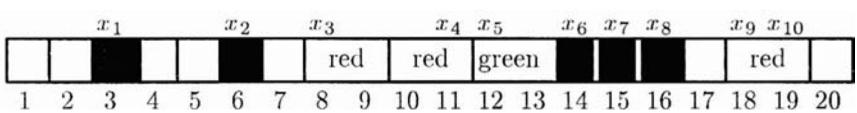

FIGURE 2. A colored tiling based on $S = \{3, 6, 8, 11, 12, 14, 15, 16, 18, 19\}$.

When t is odd, we create intervals $I_1 = [x_2, x_3]$, $I_2 = [x_4, x_5], \ldots, I_{(t-1)/2} = [x_{t-1}, x_t]$ which obey the same coloring rules as before. All cells outside these intervals are covered by a white square, except for cell x_1 which is covered by a black square. Since every interval allows 5 choices, the subset $\{x_1, \ldots, x_t\}$ gives us $5^{\lfloor t/2 \rfloor}$ ways to create a colored tiling.

The coloring rules, as stated, have two deficiencies, which conveniently complement each other. The first problem is that a string of dominoes of the same color can be generated by more than one subset. For example, the coloring in Figure 2 could also have been generated by the subset

$\{3, 6, 8, \mathbf{9}, \mathbf{10}, 11, 12, 14, 15, 16, 18, 19\}$. The other problem is that the coloring rules provide no means of generating a colored tiling such as the one in Figure 3. We remedy these problems in one fell swoop by amending our

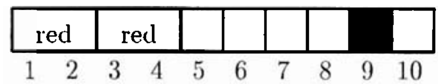

FIGURE 3. Another colored tiling.

coloring rules as follows. When an even interval $I_j = [a, b]$ tiled by dominoes immediately precedes interval $I_{j+1} = [c, d]$, i.e., $c = b + 1$, then I_{j+1} cannot be given the same color as I_j. Instead, we allow I_{j+1} to be covered by white squares ending with a single black square. Now the tiling in Figure 3 can only be obtained by the subset $\{1, 4, 5, 9\}$. Notice the amended rule still allows exactly 5 choices for each interval, and that every subset $\{x_1, \ldots, x_t\}$ leads to exactly $5^{\lfloor t/2 \rfloor}$ distinct n-tilings.

It remains to prove that every colored n-tiling is represented by exactly one subset S of $\{1, 2, \ldots, n\}$. Given an arbitrary coloring, our rules require that every cell covered by a black square must be in S, as must any cell appearing at the beginning of a "color run", (e.g., cells 8, 12, and 18 in Figure 2). Almost all cells covered by white squares are not in S. Cells covered by white squares that come immediately after the end of a color run, may or may not be in S. All cells in the interior of a color run are not in S. The only cells whose status remains to be determined are the last cells of a color run and possibly (if it's covered by a white square) the cell after that.

Notice that if no color runs exist (all squares), then S is simply the set of cells covered by the black squares. Otherwise, let j denote the last cell of our last color run (e.g., cell 19 in Figure 2 or cell 4 in Figure 3). If $j = n$, then obviously $j \in S$. Otherwise, let $B \geq 0$ denote the number of black squares occurring after cell j. If B is even, then $j \in S$, and only the black cells which follow it are in S. If B is odd, then we look at cell $j + 1$ which is necessarily covered by a square. If cell $j + 1$ is covered by a black square, then we must have $j \notin S$ and $j + 1 \in S$. If $j + 1$ is covered by a white square, then we must have $j \in S$ and $j + 1 \in S$. Thus we have uniquely determined the status of all cells from the beginning of the last color run. Ignoring these cells, we determine the status of the previous cells recursively, by the exact same argument. For example, the colored tiling in Figure 2 can only be generated by the given subset S. This completes our one-to-one correspondence, and identity (1) is proven.

Finally, we establish identity (2). After multiplying both sides by 2, we obtain:

$$2 \cdot \binom{n}{0} + 2 \cdot 5 \binom{n}{2} + 2 \cdot 5^2 \binom{n}{4} + \cdots + 2 \cdot 5^k \binom{n}{2k} + \cdots = 2^n L_n.$$

The right side counts the number of colored n-bracelets. For each even subset $\{x_1, x_2, \ldots, x_{2k}\}$, we generate $2 \cdot 5^k$ colored n-bracelets. As before, create the intervals $I_1 = [x_1, x_2]$, $I_2 = [x_3, x_4]$, $\ldots$, $I_k = [x_{2k-1}, x_{2k}]$. We generate 5^k colored tilings of an n-board by the same coloring rules as before. Gluing the ends of a tiled n-board creates an n-bracelet. In this manner we obtain all *in phase bracelets with an even number of black squares (possibly zero) before the first domino*. We call such bracelets *simple*.

By the argument in our last identity, all simple bracelets are generated by our coloring rules as k varies from 0 to $\lfloor n/2 \rfloor$. For each simple bracelet B we define a unique companion bracelet B' by conditioning on cell 1 of B. (See Figure 4.)

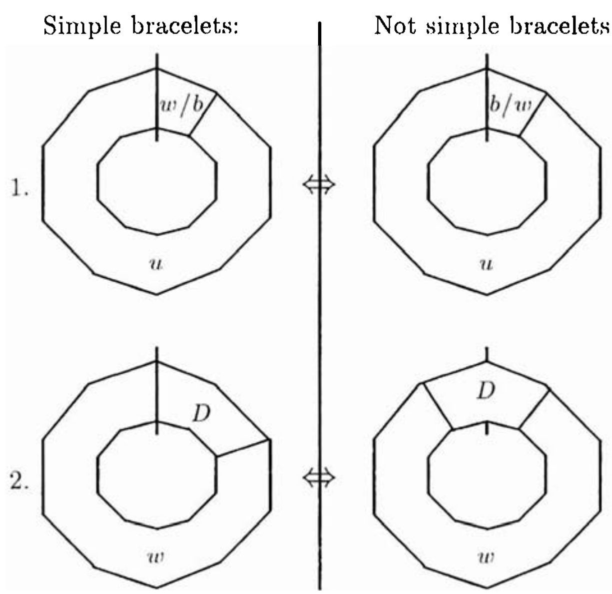

FIGURE 4. A one-to-one correspondence between simple bracelets (in phase bracelets with an even number of black squares before the first domino) and not simple ones.

1. If cell 1 of B is covered by a square, then change the color of the square covering cell 1, producing an in phase bracelet B' with an odd number of blacks squares before the first color.
2. If cell 1 of B is covered by a domino, then rotate the tiles of B counterclockwise one cell, producing an out of phase colored bracelet B'.

Since this correspondence is easily reversed, the proof of identity (2) is complete.

3. Concluding Remarks

Colored tilings provide a fresh approach to understanding Fibonacci and Lucas number relations. Perhaps this approach can be extended to envision relations involving $a^n f_n$, $a^n L_n$, or even $2^n G_n$ for generalized Fibonacci numbers G_n [2]. For further combinatorial interpretations of Fibonacci and Lucas numbers, see [1, 3, 4].

References

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