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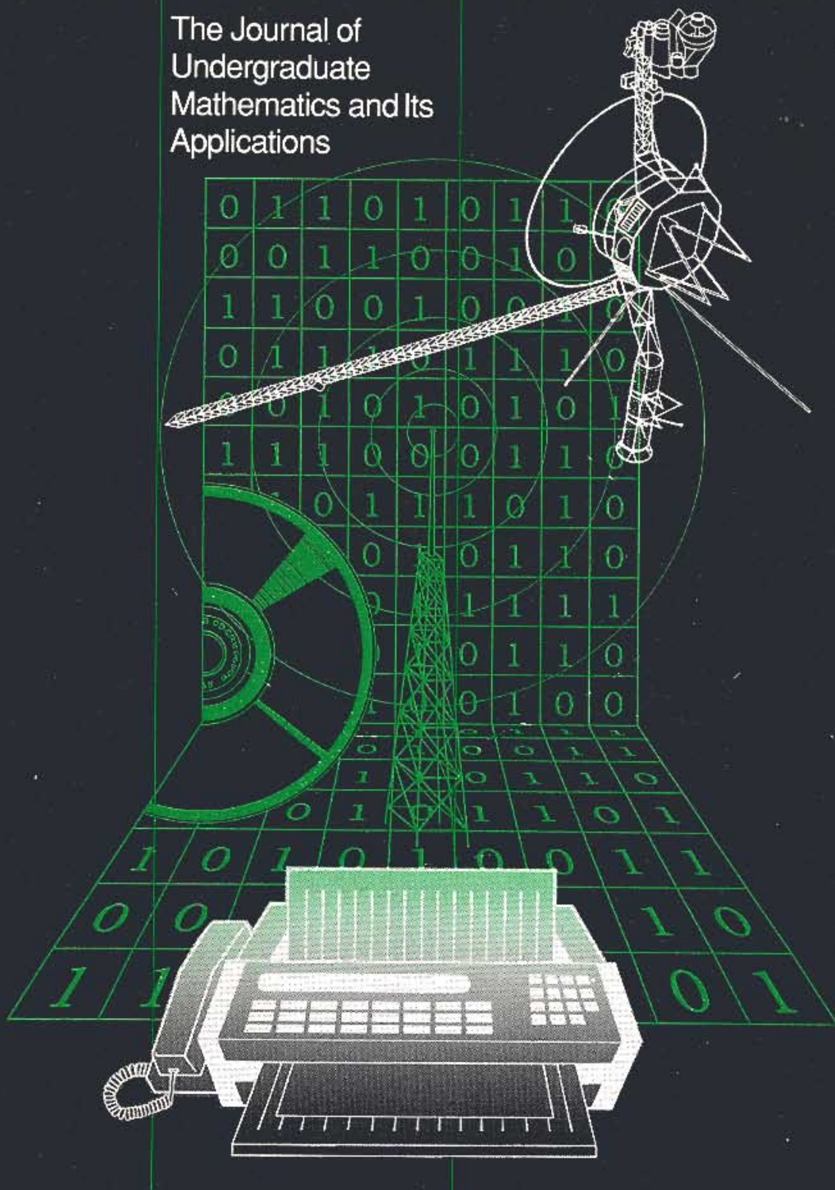

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Optimal Blackjack Strategy with "Lucky Bucks"

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Introduction

In the casino game blackjack or "21," mathematically determined best plays have been calculated by various mathematicians and gambling experts. These optimal playing strategies all assume that the casino pays even money on bets (excluding when the player has a "blackjack"). However, many casinos offer the player "lucky bucks" that pay the player either 3-to-2 or 2-to-1. In the usual game, the player's expected loss is under 1¢ per dollar bet. In this paper, we derive optimal strategies under lucky-buck conditions, giving the player an expected *gain* of 26¢ or 55¢ per dollar bet.

The Rules of the Game

The game of blackjack pits the player against the dealer ("the house"). The player is dealt two cards face up, and the dealer is dealt one card face up and one card face down. "Ten-cards" (face cards and tens) are worth 10, aces are worth 1 or 11, and all other cards are worth face value. The player's objective is to obtain a hand that is better than the dealer's ultimate hand. The best hand is called a "blackjack," which occurs when the player's original two cards are an ace and a ten-card. A blackjack pays the player 3-to-2, unless the dealer also has blackjack, in which case the player loses. For non-blackjack hands, if the sum of the player's cards does not exceed 21 and is higher than the dealer's total, then the player wins the amount bet. (Note: A blackjack beats a 21.) If the player's total exceeds 21, the player "busts" and loses the bet, even if the dealer subsequently busts (therein lies the house advantage). If the player and dealer have the same total below 22, then the hand is called a "push," and the player keeps the bet.

After the initial cards are dealt, the player (who goes first) has three (and sometimes four) options. The player can

- "stand," keeping just the two cards;
- "hit," receiving an additional card face up with the option of hitting again, or
- "double down," doubling the bet and receiving just one additional card.

In the situation where the player's first two card values are identical, the player has a fourth option, splitting, in which the player places an additional bet and plays the two cards as separate hands. After a hand is split, a blackjack counts only as 21, and doubling down is not permitted, but resplitting cards is allowed. Also, in most casinos, after splitting aces, the player is only dealt one more card to each hand. When the player has completed the turn, the dealer then hits until the house's cards total 17 or more. Thus, the dealer's strategy is fixed; and the only possible outcomes for the dealer are hands that total 17 through 21 or a bust.

If we assume that the lucky buck pays 2-to-1, then the rules are as follows. When the player uses a lucky buck, the player risks only $1 and the house matches this dollar; thus, winning hands pay off $2 and losing hands lose only $1 plus the lucky buck. Moreover, if the player chooses an option that requires placing an additional bet, the player need only bet one additional dollar, which will again be matched by the house. For example, a blackjack wins $3, and a hand that has been doubled down will now be worth $4 if it is a winner.

Methods

In order to determine the player's best strategy in any given situation, we wrote a dynamic program to compare the expected values of all possible actions under every specific circumstance. Although in reality casinos use from one to six decks, the complexity of the program was greatly simplified by assuming that the cards were being dealt from an infinite deck; thus the probability of receiving any card remained constant at 1/13. Despite this assumption, the optimal strategy for the 1-to-1 payback ratio is nearly identical to the optimal strategy derived by Uston [1981], so the strategies for the other payback ratios are assumed to be very near optimal. Optimal blackjack strategy has also been analyzed using other mathematical techniques: combinatorial analysis [Braun 1975], statistics [Griffin 1988], and computer simulation [Thorp 1966].

The dynamic program is in five parts. The program

- uses a simple recursion to calculate the dealer's probability of reaching any possible outcome from all possible initial cards,
- employs dynamic programming to determine whether hitting or standing maximizes the expectancy of any given hand,
- determines whether doubling down dominates the current best play,

- determines whether splitting (when possible) is the best option and stores results for splitting in separate arrays, and
- calculates a weighted average of the expected values from all possible initial situations, thus giving the total expected value of playing the game.

In order to determine the dealer's probabilities, we use two arrays: one for "hard" hands, in which the dealer has no aces, and another for "soft" hands, in which the dealer has at least one ace.

Now for some math! Define

$$\begin{aligned} P[i,j] &= \text{the probability that the dealer reaches outcome } i, \\ &\qquad \text{starting with a total of hard } j, \text{ and} \\ SP[i,j] &= \text{the probability that the dealer reaches outcome } i, \\ &\qquad \text{starting with a total of soft } j \\ &\qquad \text{(i.e., the cards can total } j \text{ or } j+10). \end{aligned}$$

Thus, as a base case, for $j \geq 22$, we have $P[\text{bust}, j] = 1$ and $P[i,j] = 0$ for $i \leq 21$. Likewise, for $j = 17, \ldots, 21$, we have $P[i,j] = 1$ for $i = j$, and $P[i,j] = 0$ for $i \neq j$. In general, the recursions are as follows for $i \geq 17$:

$$P[i,j] = \frac{1}{13} \sum_{k=2}^{9} (P[i,j+k] + 4P[i,j+10] + SP[i,j+1]), \quad j \geq 2;$$

$$SP[i,j] = \begin{cases} P[i,j], & j \geq 12; \\ P[i,j+10], & 7 \leq j \leq 11; \\ \frac{1}{13}\Sigma_{k=1}^{9} (SP[i,j+k] + 4SP[i,j+10]), & j \leq 6. \end{cases}$$

The first and last equations are calculated by conditioning on what the next card will be and applying the law of total probability. Also, in the actual computer program, care must be taken that these calculations are performed in a specific order. For example, to calculate $P[18,5]$, we must know $SP[18,6]$, in case the dealer draws an ace. If the dealer is dealt a blackjack, the player automatically loses the bet unless the player too has a blackjack, in which case the hand is a push. Thus, if the dealer has an ace or a ten-card showing and does not have a blackjack, the player should exploit this information. Toward that end, we shall redefine

$$\begin{aligned} P[i,j] &= \text{the probability that the dealer reaches } i \text{ from a total of } j, \\ &\qquad \text{given that the dealer does not have blackjack.} \end{aligned}$$

For $j = 2, \ldots, 9$, the value of $P[i,j]$ is unchanged. However, if the dealer shows a ten-card or an ace, then the blackjack probability is $1/13$ or $4/13$. Hence, by conditional probability, we make the following reassignments:

$$P[21,10] \leftarrow \frac{P[21,10] - \frac{1}{13}}{\frac{12}{13}}$$

$$
\begin{aligned}
&= \frac{13}{12} P[21,10] - \frac{1}{12}, \\
P[i,10] &\leftarrow \frac{13}{12} P[i,10] \qquad \text{for } i \neq 21; \\
P[21,\text{ace}] &= \frac{13}{9} P[21,\text{ace}] - \frac{4}{9}, \\
P[i,\text{ace}] &= \frac{13}{9} P[i,\text{ace}] \qquad \text{for } i \neq 21.
\end{aligned}
$$

Knowing the above probabilities, we are able to determine the player's optimal strategy. Let $E[x,y]$ = the player's expected profit of currently having a hard total of y when the dealer shows an x under the optimal strategy and does not have a blackjack. Let $SE[x,y]$ be the same for soft totals. Thus, in the base case, $E[x,y] = SE[x,y] = -1$ for all $y \geq 22$.

Now define $E_a[x,y]$ and $SE_a[x,y]$ as the expected value of taking action a followed by the optimal strategy for hard and soft totals of y. The action a will either be a (*h*)it, a (*s*)tand, a (*d*)ouble down, or a s(*p*)lit. Before we determine the expected values let us define

$$
\begin{aligned}
F[x,y] &= \max(E_s[x,y], E_h[x,y]), \\
SF[x,y] &= \max(SE_s[x,y], SE_h[x,y]).
\end{aligned}
$$

Thus, if we wager \$1 with payback ratio r, we have for $x = \text{ace}, 2, \ldots, 10$,

$$
\begin{aligned}
E_s[x,y] &= r\left(P[\text{bust},x] + \sum_{k=17}^{y-1} P[k,x]\right) - \sum_{k=y+1}^{21} P[k,x], \\
SE_s[x,y] &= \begin{cases} E_s[x,y], & y \geq 12; \\ E_s[x,y+10], & y \leq 11. \end{cases}
\end{aligned}
$$

The first equation states that if the player stands with a y against the dealer's x, the player will win \$$r$ if the dealer's ultimate total is below x or the dealer busts, and will lose \$1 if the dealer's ultimate total beats y.

$$
\begin{aligned}
E_h[x,y] &= \frac{1}{13} \sum_{k=2}^{9} \left(F[x,y+k] + 4F[x,y+10] + SF[x,y+1]\right), \\
SE_h[x,y] &= \frac{1}{13} \sum_{k=1}^{9} \left(SF[x,y+k] + 4SF[x,y+10]\right).
\end{aligned}
$$

The Fs and SFs in the above equations reflect the fact that once the player hits, splitting and doubling down are no longer options.

$$
\begin{aligned}
E_d[x,y] &= \frac{2}{13} \sum_{k=2}^{9} \left(E_s[x,y+k] + 4E_s[x,y+10] + SE_s[x,y+1]\right), \\
SE_d[x,y] &= \frac{2}{13} \sum_{k=1}^{9} \left(SE_s[x,y+k] + 4SE_s[x,y+10]\right),
\end{aligned}
$$

since the bet has been doubled and we must stand after the next card. If y is the sum of two aces,

$$E_p[x,2] = \frac{2}{13}\sum_{k=1}^{9}(SE_s[x,1+k] + 4SE_s[x,11])\,;$$

if y is the sum of two ten-cards,

$$E_p[x,20] = \frac{2}{13}\sum_{k=2}^{9}(F[x,10+k] + SF[x,11] + 4E_p[x,20])\,.$$

The last E_p term reflects the possibility that we draw another ten-card and resplit again. Solving for E_p, we get

$$\begin{aligned} E_p[x,20]\left(1-\frac{8}{13}\right) &= \frac{2}{13}\sum_{k=2}^{9}(F[x,10+k] + SF[x,11])\,, \\ E_p[x,y] &= \frac{2}{5}\sum_{k=2}^{9}(F[x,10+k] + SF[x,11])\,. \end{aligned}$$

Otherwise, if the total y consists of two $y/2$ cards, we have

$$E_p[x,y] = \frac{2}{13}\sum_{\substack{k=2 \\ k\neq y/2}}^{9}(F[x,y/2+k] + SF[x,y/2+1] + 4F[x,y/2+10] + E_p[x,y])\,,$$

which leads to

$$E_p[x,y] = \frac{2}{11}\sum_{\substack{k=2 \\ k\neq y/2}}^{9}(F[x,y/2+k] + SF[x,y/2+1] + 4F[x,y/2+10])\,.$$

At this point, all of the expected values have been calculated based on the assumption that the dealer did not have a blackjack. In order to include this further possibility, we make the following adjustments for all actions $a = (s,h,d,p)$ and for all y:

$$\begin{aligned} E_a[10,y] &\leftarrow \frac{12}{13}\,E_a[10,y] - \frac{1}{13}, \\ E_a[\text{ace},y] &\leftarrow \frac{9}{13}\,E_a[\text{ace},y] - \frac{4}{13}. \end{aligned}$$

The exception to this is when the player's hand is also a blackjack, when

$$\begin{aligned} E_s[10,\text{bj}] &= \frac{12}{13}\times\frac{3}{2}\times r, \\ E[\text{ace},\text{bj}] &= \frac{9}{13}\times\frac{3}{2}\times r, \qquad \text{and} \\ E_s[x,\text{bj}] &= \frac{3}{2}\times r, \end{aligned}$$

for $x = 2, \ldots, 9$. Now the program computes the maximum of all possible options $E[x,y]$ and $SE[x,y]$:

$$E[x,y] = \begin{cases} \max(E_s[x,y], E_h[x,y], E_d[x,y], E_p[x,y]), & \text{if } y \text{ can be split;} \\ \max(E_s[x,y], E_h[x,y], E_d[x,y]), & \text{otherwise.} \end{cases}$$

$$SE[x,y] = \max(SE_s[x,y], SE_h[x,y], SE_d[x,y]).$$

Now that all the expectancies and optimal actions are known, the program computes the weighted average of all $13^2 = 169$ possible starting hands for the player versus the dealer's ten possible upcards. This number is the expected value of the game for one hand.

Results

Table 1 indicates the dealer's probabilities given that the dealer does not have a blackjack, and **Table 2** indicates the results of the final program for three different values of r. The left column indicates the player's current total and the row across the top indicates the dealer's upcard ("0" stands for 10), with H standing for (H)it, S for (S)tand, D for (D)ouble down, and P for s(P)lit. Notice that for a payback ratio of 1, the player rarely uses the double-down option and splits mainly on aces, 8s and 9s. Two good cases to check in the 1-to-1 game are whether the player hits a 12 versus a 2 or 3 but not versus a 4, 5, or 6, and whether the player splits 9s versus 2 through 9 except 7. Fortunately, these results hold.

It can be seen from **Table 2** that as the payback ratio r increases, the player begins to double down and split more often. In these situations, placing an additonal bet means that the house matches this additional bet with an invisible lucky buck. Therefore, we expected some optimal plays to change from hits or stands to double downs or splits.

Interesting Results

On the other hand, we did not expect the results in the situation in which the player has 16 and the dealer has a 10. In the 1-to-1 game, the best play is a hit; but in the 1.5-to-1 and 2-to-1 games, the best play is a stand. Our original conjecture was that no optimal plays would change from hits to stands, or from stands to hits, because no additional money would be risked and no additional lucky bucks would be gained. However, this result can be explained by the algebra below.

Let P_{oa} denote the player's probability of a specific outcome o (winning, losing, or pushing) by taking a certain action a (hitting or standing). Then E_{ar} (the expected value of taking action a in the game with payback ratio r)

Table 1.
Dealer's probabilities (no blackjack).

Hard hands:	17	18	19	20	21	bust
4	.130	.126	.121	.116	.111	.394
5	.122	.122	.118	.113	.108	.416
6	.165	.106	.106	.102	.097	.423
7	.369	.138	.079	.079	.074	.262
8	.129	.359	.129	.069	.069	.245
9	.120	.120	.351	.120	.061	.228
10	.121	.121	.121	.371	.037	.230
11	.111	.111	.111	.111	.342	.212
12	.103	.103	.103	.103	.103	.483
13	.096	.096	.096	.096	.096	.520
14	.089	.089	.089	.089	.089	.554
15	.083	.083	.083	.083	.083	.586
16	.077	.077	.077	.077	.077	.615
17	1	0	0	0	0	0
18	0	1	0	0	0	0
19	0	0	1	0	0	0
20	0	0	0	1	0	0
21	0	0	0	0	1	0

Soft hands:	17	18	19	20	21	bust
S1	.189	.189	.189	.189	.078	.167
S2	.151	.151	.151	.151	.151	.245
S3	.146	.146	.146	.146	.146	.272
S4	.140	.140	.140	.140	.140	.300
S5	.135	.135	.135	.135	.135	.327
S6	.129	.129	.129	.129	.129	.354
S7	1	0	0	0	0	0
S8	0	1	0	0	0	0
S9	0	0	1	0	0	0
S10	0	0	0	1	0	0
S11	0	0	0	0	1	0

Table 2.
Optimal strategies with playback ratios 1, 1.5, and 2.

	Payback ratio = 1	Payback ratio = 1.5	Payback ratio = 2
Hard hands:	A234567890	A234567890	A234567890
2	HHHHHHHHHH	HHHHHHHHHH	HHHHHHHHHH
3	HHHHHHHHHH	HHHHHHHHHH	HHHHDDHHHH
4	HHHHHHHHHH	HHHHHHHHHH	HHHHDDHHHH
5	HHHHHHHHHH	HHHHHHHHHH	HHHHDDHHHH
6	HHHHHHHHHH	HHHHHHHHHH	HHHDDDHHHH
7	HHHHHHHHHH	HHHHHDHHHH	HHDDDDHHHH
8	HHHHHHHHHH	HDDDDDHHHH	HDDDDDDHHH
9	HHDDDDHHHH	DDDDDDDDDD	HDDDDDDDDD
10	HDDDDDDDDH	DDDDDDDDDD	DDDDDDDDDD
11	HDDDDDDDDD	DDDDDDDDDD	DDDDDDDDDD
12	HHHSSSHHHH	HHHSSSHHHH	HDDDDDDHHH
13	HSSSSSHHHH	HSSSSSHHHH	HSSSSSDHHH
14	HSSSSSHHHH	HSSSSSHHHH	HSSSSSHHHH
15	HSSSSSHHHH	HSSSSSHHHH	HSSSSSHHHH
16	HSSSSSHHHH	HSSSSSHHHS	HSSSSSHHHS
17	SSSSSSSSSS	SSSSSSSSSS	SSSSSSSSSS
18	SSSSSSSSSS	SSSSSSSSSS	SSSSSSSSSS
19	SSSSSSSSSS	SSSSSSSSSS	SSSSSSSSSS
20	SSSSSSSSSS	SSSSSSSSSS	SSSSSSSSSS
21	SSSSSSSSSS	SSSSSSSSSS	SSSSSSSSSS
Soft hands:	A234567890	A234567890	A234567890
2	HHHHHHHHHH	HDDDDDHHHH	HDDDDDHHHH
3	HHHHHDHHHH	HDDDDDHHHH	HDDDDDDHHH
4	HHHHDDHHHH	HDDDDDHHHH	HDDDDDDDHH
6	HHHDDDHHHH	HDDDDDDHHH	HDDDDDDDDD
7	HHDDDDHHHH	HDDDDDDDHH	HDDDDDDDDD
8	HSDDDDSSHH	HDDDDDDDHH	DDDDDDDDDD
9	SSSSSSSSSS	SDDDDDSSSS	SDDDDDDSSD
10	SSSSSSSSSS	SSSSDDSSSS	SDDDDDSSSS
11	SSSSSSSSSS	SSSSSSSSSS	SSSSSSSSSS
Split:	A234567890	A234567890	A234567890
1	PPPPPPPPPP	PPPPPPPPPP	PPPPPPPPPP
2	HHHPPPPHHH	PPPPPPPPPH	PPPPPPPPPP
3	HHHPPPPHHH	PPPPPPPPPH	PPPPPPPPPP
4	HHHHDDHHHH	HDDDPDHHHH	PPPPPDPPPH
5	HDDDDDDDDH	DDDDDDDDDD	DDDDDDDDDD
6	HHPPPPHHHH	HPPPPPPPPP	PPPPPPPPPP
7	HPPPPPPHHH	HPPPPPPPPP	PPPPPPPPPP
8	PPPPPPPPPP	PPPPPPPPPP	PPPPPPPPPP
9	SPPPPPSPPS	PPPPPPPPPP	PPPPPPPPPP
10	SSSSSSSSSS	SSSSSSSSSS	SPPPPPPSSS

for the two actions are:

$$\begin{aligned} E_{s1} &= P_{ws}(1) + P_{ls}(-1) + P_{ps}(0) = P_{ws} - P_{ls} \\ E_{h1} &= P_{wh} - P_{lh} \\ E_{s2} &= 2P_{ws} - P_{ls} \\ E_{h2} &= 2P_{wh} - P_{lh}. \end{aligned}$$

The way to understand how $E_{h1} > E_{s1}$, but $E_{h2} < E_{s2}$, is to look at the value of the push. By standing, the probability of pushing is 0, since the dealer never stops on a 16. However, by hitting, the probability of pushing is greater than 0. Thus:

$$\begin{aligned} P_{ls} &= 1 - P_{ws} \\ P_{lh} &= 1 - P_{wh} - P_{ph}. \end{aligned}$$

Plugging these equations into the expected-value formulas above, we get:

$$\begin{aligned} E_{s1} &= 2P_{ws} - 1 \\ E_{h1} &= 2P_{wh} + P_{ph} - 1 \\ E_{s2} &= 3P_{ws} - 1 \\ E_{h2} &= 3P_{wh} + P_{ph} - 1. \end{aligned}$$

Thus:

$$2P_{wh} + P_{ph} > 2P_{ws} \qquad \text{but} \qquad 3P_{wh} + P_{ph} < 3P_{ws},$$

so

$$2P_{ws} < 2P_{wh} + P_{ph} < 3P_{wh} + P_{ph} < 3P_{ws}.$$

Although this situation is not true for all P_{oa} values, it is true for some small range of values. Hence, we conclude that this unexpected result is not contradictory.

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