NOTES

Paint It Black—A Combinatorial Yawp

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Have you experienced a “mathematical yawp” lately? (Not sure you want to answer until you know what one is?) Well, the phrase “mathematical yawp” was coined by Francis Su in his James R. Leitzel Lecture at the 2006 MathFest. In essence, a mathematical yawp is one of those “light bulb” or “aha!” moments when a mathematician comes to an understanding of a topic so moving that it is accompanied by a yelp of joy or disbelief. By specialization, a *combinatorial yawp* is one of those moments achieved while counting.

Combinatorial proofs are appreciated for the elegance and/or simplicity of their arguments (see [**2**]). However, the true (and frequently underappreciated) beauty lies in their power to generalize results. Understanding the components of a mathematical identity in a concrete counting context provides the first clue for exploring natural extensions. Investigating and stretching the role of each parameter in turn, leads to different generalizations—ones that might not be connected without the combinatorial insight.

Our yawp occurred while exploring Problem # 11220, proposed by David Beckwith, from the April 2006 issue of the *American Mathematical Monthly* [**1**], the innocuous-looking alternating binomial identity below.

IDENTITY 1. *For* $n \geq 1$,

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{n-1} = 0.$$

Equipped with the ability to select subsets, to paint elements black, blue, or white, and to count, we will work through a novel proof of this identity and then explore numerous related results. What qualifies as a natural generalization is open to debate, but the greatest surprise is the sheer number of interesting generalizations to be explored.

To prove Identity 1, begin by understanding the unsigned quantity in the alternating sum, $\binom{n}{r}\binom{2n-2r}{n-1}$. Consider the set of n consecutive pairs, $\{\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}\}$. Given r, $0 \le r \le n$, select r of the **pairs** to paint black in $\binom{n}{r}$ ways. Of the remaining $2n-2r$ elements that have not yet been painted, select $n-1$ to paint blue. The remaining elements are then painted white. We call such a painted set a *configuration*. For example, when $n = 5$,

$$X = \{\{1, \underline{2}\}, \{\underline{3}, \underline{4}\}, \{\mathbf{5}, \mathbf{6}\}, \{\underline{7}, 8\}, \{\mathbf{9}, \mathbf{10}\}\}$$

is a configuration where black elements are bold, blue elements are underlined, and the remaining elements are white.

Now define two sets, denoted $\mathcal{E}$ and $\mathcal{O}$, that depend on the parameter r, the number of black pairs.

Set $\mathcal{E}$. All configurations with an even number of black pairs.

Set $\mathcal{O}$. All configurations with an odd number of black pairs.

Since a configuration from $\mathcal{E}$ contributes $+1$ to the summation while a configuration from $\mathcal{O}$ contributes -1, the left-hand side of Identity 1 is simply $|\text{Set } \mathcal{E}| - |\text{Set } \mathcal{O}|$. If we can show that $|\text{Set } \mathcal{E}| = |\text{Set } \mathcal{O}|$, then Identity 1 is proved. Our goal then is to find a bijection between $\mathcal{E}$ and $\mathcal{O}$.

Correspondence. Find the minimum integer j such that $1 \le j \le n$ and $\{2j-1, 2j\}$ contains no blue element, i.e., it is either a black pair or a white pair. Then toggle the color of this pair—if it is black, make it white and if it is white, make it black.

Since there are only $n-1$ blue elements (and n total pairs), every configuration has at least one pair containing no blue element. So j always exists and the correspondence is a bijection. Hence, $|\text{Set } \mathcal{E}| = |\text{Set } \mathcal{O}|$ and the proof is complete.

As an illustration, the previously considered configuration

$$X = \{\{1, \underline{2}\}, \{\underline{3}, \underline{4}\}, \{\mathbf{5}, \mathbf{6}\}, \{\underline{7}, 8\}, \{\mathbf{9}, \mathbf{10}\}\},$$

belongs to $\mathcal{E}$ since it contains $r = 2$ black pairs. By toggling the first blueless pair $\{5, 6\}$, X is matched with

$$X' = \{\{1, \underline{2}\}, \{\underline{3}, \underline{4}\}, \{5, 6\}, \{\underline{7}, 8\}, \{\mathbf{9}, \mathbf{10}\}\},$$

which belongs to $\mathcal{O}$, since it has $r = 1$ black pair.

At this point, many natural questions arise. Can we change the number of blue elements? What happens if we replace the *pairs* above by *k-sets*? Can we say something about partial sums? We will consider each of these questions in turn.

Changing the number of blue elements. If we paint fewer than $n-1$ elements blue in our proof above, the argument doesn't change. We are still guaranteed a blueless pair, so a toggle point exists. Letting m represent the number of blue elements to be painted, this gives

IDENTITY 2. *For* $0 \le m < n$,

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{m} = 0.$$

What happens when m is larger than $n-1$? Well, the initial set-up is the same. Select r pairs to color black and m of the remaining elements to color blue. The sets $\mathcal{E}$ and $\mathcal{O}$ contain configurations with an even or odd number of black pairs. Again, toggle the color of the first blueless pair. Unfortunately, there are now unpaired configurations in our correspondence (so it is no longer a bijection). Since m is greater than or equal to n, we can no longer guarantee a toggle point exists. However, we know that the unpaired configurations have at least one blue element in every pair, so these configurations have zero black pairs and hence belong to $\mathcal{E}$.

For example, when $n = 5$ and $m = 7$, the configuration

$$X = \{\{\underline{1}, 2\}, \{\underline{3}, \underline{4}\}, \{\underline{5}, \underline{6}\}, \{\underline{7}, 8\}, \{9, \underline{10}\}\}$$

has no toggle point.

How many of these unpaired configurations are there? Such configurations have $m-n$ pairs where both elements are painted blue. So there are $\binom{n}{m-n}$ ways to select the blue pairs. Then, the other $n-(m-n) = 2n-m$ pairs have one blue element and one white element, and there are 2^{2n-m} ways to paint them. Thus, there are $\binom{n}{m-n}2^{2n-m}$ unpaired configurations, leading to our next generalization.

IDENTITY 3. *For* $n, m \ge 0$,

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{2n-2r}{m} = 2^{2n-m}\binom{n}{m-n}.$$

Note that this is a generalization of Identity 2 since $\binom{n}{m-n} = 0$ when $m < n$. To some, this would be enough for a yawp. But we press on for more!

From pairs to *k*-sets. Rather than creating n subsets by pairing consecutive elements of the set $\{1, 2, 3, \ldots, 2n\}$, we ask what would happen if we group k consecutive elements from $\{1, 2, 3, \ldots, kn\}$. By mimicking the argument for Identity 1, we can immediately generalize Identity 2 as follows.

IDENTITY 4. *For* $0 \le m < n$ *and* $k \ge 1$,

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}\binom{kn-kr}{m} = 0.$$

For example, when $n = 5$, $k = 3$, $m = 4$, the configuration

$$X = \{\{\underline{1}, 2, \underline{3}\}, \{\mathbf{4}, \mathbf{5}, \mathbf{6}\}, \{7, 8, 9\}, \{\underline{10}, \underline{11}, 12\}, \{\mathbf{13}, \mathbf{14}, \mathbf{15}\}\}$$

has $r = 2$ black 3-sets (and thus belongs to $\mathcal{E}$) and by toggling the first blueless 3-set, we get

$$X' = \{\{\underline{1}, 2, \underline{3}\}, \{4, 5, 6\}, \{7, 8, 9\}, \{\underline{10}, \underline{11}, 12\}, \{\mathbf{13}, \mathbf{14}, \mathbf{15}\}\}$$

(which belongs to $\mathcal{O}$).

Can we generalize Identity 4, allowing $m \ge n$ blue elements? Yes and no. We can formulate a general answer, but the alternating sum becomes a sum over integer partitions. Although it is not the nice answer we were hoping for, it still has some notable specializations.

In the general situation with $m \geq n$, unpaired objects are configurations with at least one blue element in every k-set. These objects necessarily belong to $\mathcal{E}$ since they have $r = 0$ black k-sets. For example, when $n = 5$, $k = 3$, $m = 8$, the configuration

$$\{\{\underline{1}, 2, 3\}, \{\underline{4}, \underline{5}, \underline{6}\}, \{\underline{7}, 8, 9\}, \{\underline{10}, 11, \underline{12}\}, \{13, \underline{14}, 15\}\}$$

has no blueless 3-set. We can count these by considering the distribution of m blue elements among the n different k-sets. Let x_i count the number of k-sets containing i blue elements $(1 \leq i \leq k)$. In our example, $x_1 = 3$, $x_2 = 1$, $x_3 = 1$. The sum $\sum_{i=1}^{k} x_i$ counts the number of k-sets containing blue elements while the sum $\sum_{i=1}^{k} i x_i$ counts the number of blue elements. Only nonnegative integer solutions $(x_1, x_2, \ldots, x_n)$ to

$$\begin{cases} n = x_1 + x_2 + \cdots + x_k \\ m = x_1 + 2x_2 \cdots + kx_k \end{cases}$$

contribute to the number of unpaired configurations. Since the number of ways to choose which x_i k-sets have i blue elements is the multinomial coefficient

$$\binom{n}{x_1, x_2, \ldots, x_k} = \frac{n!}{x_1!\, x_2! \cdots x_k!},$$

and a k-set with i blue elements can be painted $\binom{k}{i}$ ways, we get

IDENTITY 5. *For all $k, m, n \geq 1$,*

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \binom{kn - kr}{m} = \sum_{(x_1, x_2, \ldots, x_k)} \binom{n}{x_1, x_2, \ldots, x_k} \prod_{i=1}^{k} \binom{k}{i}^{x_i},$$

where the sum on the right is taken over all simultaneous nonnegative integer solutions to $n = x_1 + x_2 + \cdots + x_k$ and $m = x_1 + 2x_2 \cdots + kx_k$.

Note that this is a generalization of Identity 4 since when m is less than n, the sum on the right is empty. Some special cases are worth mentioning because their right-hand sides reduce to simple one-term expressions:

- $m = n$

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \binom{kn - kr}{n} = k^n.$$

- $m = n + 1$

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} \binom{kn - kr}{n + 1} = nk^{n-1} \binom{k}{2}.$$

Partial sums. The final generalization considers what happens if we return to creating pairs from the set $\{1, 2, 3, \ldots, 2n\}$ and only consider the first s terms of the original sum. To make life easier, we restrict our attention to the situation where $m < n$ and consider

$$\sum_{r=0}^{s} (-1)^r \binom{n}{r} \binom{2n - 2r}{m}.$$

In this case, the development parallels Identity 2 except that only configurations with s or fewer black pairs are considered. To match configurations between $\mathcal{E}$ and $\mathcal{O}$, we toggle the color of the first blueless pair unless the configuration contains the maximum s black pairs and a white pair precedes them.

For example, when $n = 5$, $m = 2$, $s = 3$ the configuration

$$X = \{\{1, 2\}, \{\mathbf{3}, \mathbf{4}\}, \{\mathbf{5}, \mathbf{6}\}, \{\underline{7}, \underline{8}\}, \{\mathbf{9}, \mathbf{10}\}\},$$

is unmatched, since by toggling the first blueless set $\{1, 2\}$, we would wind up with four black pairs, exceeding our upper bound. We note that among the configurations with s black pairs and w white pairs, the fraction of those where a white pair comes before a black pair is $\frac{w}{w+s}$.

To count the number of unmatched objects, let b represent the number of blue pairs in a configuration. Since b blue pairs contain $2b$ blue elements, there must be $m - 2b$ pairs containing one blue and one white element (and since we have s black pairs, there are $n - b - (m - 2b) - s = n - m - s + b$ white pairs). So there are $2^{m-2b}\binom{n}{s,\ b,\ m-2b,\ n-m-s+b}$ configurations with s black pairs, b blue pairs, and a total of m blue elements. Of these, $\frac{n-m-s+b}{n-m+b}$ of the configurations have a white pair coming before all the black pairs. These unmatched configurations all belong to $\mathcal{E}$ or all belong to $\mathcal{O}$ depending on the parity of s. This yields the following identity:

IDENTITY 6. *For* $0 \le m < n$ *and* $0 \le s \le n$,

$$\sum_{r=0}^{s}(-1)^r\binom{n}{r}\binom{2n-2r}{m}$$
$$= (-1)^s \sum_{b\ge 0} \frac{n-m-s+b}{n-m+b} 2^{m-2b} \binom{n}{s,\ b,\ m-2b,\ n-m-s+b}.$$

Perhaps you don't find this solution satisfactory? Let's make one last restriction in hopes of finding a "nice" solution. Restrict the location of the black pairs to only occur in the first s positions. Then, for $1 \le m, n$, the alternating sum becomes

$$\sum_{r=0}^{s}(-1)^r\binom{s}{r}\binom{2n-2r}{m}.$$

The unsigned quantity in the alternating sum, $\binom{s}{r}\binom{2n-2r}{m}$, counts the ways to select r black pairs from $\{\{1, 2\}, \{3, 4\}, \ldots, \{2s - 1, 2s\}\}$ and then paint m of the remaining uncolored elements from $\{1, 2, 3, \ldots, 2n\}$ blue. We then use the same toggling argument as before:

Set $\mathcal{E}$. All configurations with an even number of black pairs.

Set $\mathcal{O}$. All configurations with an odd number of black pairs.

Correspondence. Find the minimum integer j such that $1 \le j \le s$ and $\{2j - 1, 2j\}$ contains no blue element, i.e., it is either a black pair or a white pair. Then toggle the color of the pair.

The solutions to this alternating sum depends on the size of m, the number of blue elements to be painted. If $m < s$, a toggle point always exists and our correspondence is a bijection, giving the following generalization of Identity 2.

IDENTITY 7. *For* $0 \le m < s \le n$,

$$\sum_{r=0}^{s}(-1)^r\binom{s}{r}\binom{2n-2r}{m} = 0.$$

If $m = s$, the unmatched configurations are those in which each of the first s pairs contains at least one blue element. (Unlike the previous situation, we don't have to worry about generating too many black pairs.) All 2^s of these unmatched configurations belong to $\mathcal{E}$, and we get

IDENTITY 8. *For* $0 \le s \le n$,

$$\sum_{r=0}^{s}(-1)^r\binom{s}{r}\binom{2n-2r}{s} = 2^s.$$

Lastly, if $m > s$, the unmatched configurations are again those in which each of the first s pairs contains at least one blue element. We convert the alternating sum into a positive sum by counting the configurations that are unmatched by the previous correspondence. Such unmatched configurations have at least one blue element among each of the first s pairs (and therefore have zero black elements). For $0 \le w \le s$, we claim that there are $\binom{s}{w}\binom{2n-s-w}{m-s}$ unmatched configurations where w of the first s pairs begin with a white element. To see this, note that once we choose which s pairs begin with a white element (which can be done $\binom{s}{w}$ ways) then those w pairs must end with a blue element and the remaining $s - w$ pairs must begin with a blue element. The remaining $m - s$ blue elements can be chosen among the unspecified $(s - w) + (2n - 2s) = 2n - s - w$ elements in $\binom{2n-s-w}{m-s}$ ways. Since all of the unmatched configurations belong to $\mathcal{E}$, we arrive at our final identity, which actually encapsulates Identities 7 and 8 too.

IDENTITY 9. *For all* $m, n, s \ge 0$,

$$\sum_{r=0}^{s}(-1)^r\binom{s}{r}\binom{2n-2r}{m} = \sum_{w=0}^{s}\binom{s}{w}\binom{2n-s-w}{m-s}.$$

So starting from a single alternating binomial identity, a concrete counting context, and a good correspondence, eight related identities were explored by manipulating the roles of the parameters (and sometimes introducing new ones). The resulting identities were often beautiful generalizations—though occasionally the results didn't quite qualify as "simple" or "nice." Regardless, the questions were worth asking, the answers worth exploring, and the connections worth making. We yawped. Did you?

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