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WINNIPEG, CANADA

PATHS TO A MULTINOMIAL INEQUALITY

ARTHUR T. BENJAMIN AND JENNIFER J. QUINN

ABSTRACT. Using path counting arguments, we prove

$$\min\left\{\binom{(x_1+x_2+y_1+y_2)}{x_1,x_2,(y_1+y_2)},\binom{(x_1+x_2+y_1+y_2)}{(x_1+x_2),y_1,y_2}\right\}\leq \binom{x_1+y_1}{x_1}\binom{x_1+y_2}{x_1}\binom{x_2+y_1}{x_2}\binom{x_2+y_2}{x_2}.$$

This inequality, motivated by graph coloring considerations, has an interesting geometric interpretation.

For a rectangular lattice of size $X\times Y$, the number of increasing lattice paths between antipodal points is $\binom{X+Y}{X}$. "Folding" the lattice along the line $x=x_1$ induces a box (see Figure 1) where the number of increasing 3-dimensional lattice paths between antipodal points is $\binom{X+Y}{x_1,\ X-x_1,\ Y}=\frac{(X+Y)!}{x_1!(X-x_1)!Y!}$. Similarly, folding the original lattice along the line $y=y_1$ induces a box with $\binom{X+Y}{X,\ y_1,\ Y-y_1}$ increasing paths. We prove that at least one of these multinomial coefficients is less than or equal to the number of ways to select increasing paths from the four quadrants created by the folds. Specifically, we prove the multinomial inequality,

$$\min\left\{\binom{X+Y}{x_1,\ x_2,\ y_1+y_2},\binom{X+Y}{x_1+x_2,\ y_1,\ y_2}\right\}\leq \binom{x_1+y_1}{x_1}\binom{x_1+y_2}{x_1}\binom{x_2+y_1}{x_2}\binom{x_2+y_2}{x_2},$$

where $X=x_1+x_2$ and $Y=y_1+y_2$.

The presence of the minimum is unusual but necessary because neither term on the left-hand side satisfies the inequality by itself. Although the previous discussion may suggest a 3-dimensional approach, our proof counts paths inside a 2-dimensional lattice. We begin by proving the binomial inequality,

Lemma 1. *For nonnegative integers* x_1, x_2, y_1, *and* y_2,

$$\binom{x_1+x_2+y_1+y_2}{x_1+x_2} \leq \binom{x_1+y_1}{x_1}\binom{x_1+y_2}{x_1}\binom{x_2+y_1}{x_2}\binom{x_2+y_2}{x_2}$$

Proof. Let $R = \binom{x_1+y_1}{x_1}\binom{x_1+y_2}{x_1}\binom{x_2+y_1}{x_2}\binom{x_2+y_2}{x_2}$. Using Figure 2, $\binom{x_1+x_2+y_1+y_2}{x_1+x_2}$ counts the number of increasing paths from A to E, while R counts the number of ways to create increasing paths from A to O, B to D, H to F, and O to E simultaneously. We establish the inequality by creating a one-to-one function from the first set of paths to the second.

Suppose P is an increasing path from A to E. Let i and j respectively denote the first and last points of P which are on the boundary between two quadrants. (See Figure 3.) If i is in quadrant II, we create four increasing quadrant paths P_1, P_2, P_3, and P_4 in the following manner: the path P_3 begins at A, follows the original path P to i, and then proceeds directly to O; P_2 begins at B, proceeds directly to i, follows P until it reaches either j or O, and then proceeds directly to D; P_1 begins at O, proceeds directly to j and follows P to E; P_4 begins at H, proceeds directly to G and then directly to F. If i is in quadrant IV, the paths P_1 and P_3 are described the same way; P_4 begins at H, proceeds directly to i, follows P until it reaches either j or O, and then proceeds directly to F; P_2 begins at B, proceeds directly to C and then directly to D. The mapping is one-to-one since given P_1, P_2, P_3, and P_4 from the construction above, we can easily reconstruct P. $\square$

In the preceding proof, all paths with i in quadrant II (IV) are mapped to the same P_4 (P_2). Hence we can strengthen the inequality by introducing a "companion path", P', to accompany P in the unutilized quadrant. Specifically, if i is in quadrant II (IV), let the companion path P' be any increasing path from H to F (B to D). Thus any pair of paths P and P' can be mapped to four increasing quadrant paths where the path in the unutilized quadrant is replaced by P'. The remaining three paths are defined exactly as before. This mapping is one-to-one. Thus if we let n_q denote the number of increasing paths from A to E where i lies on the border of quadrants III and q, it follows that $n_2\binom{x_2+y_1}{x_2} + n_4\binom{x_1+y_2}{x_1} \leq R$.

Since $n_2 + n_4 = \binom{x_1+x_2+y_1+y_2}{x_1+x_2}$, we have

Corollary 2. *For nonnegative integers* $x_1,\ x_2,\ y_1,$ *and* $y_2,$

$$\binom{x_1+x_2+y_1+y_2}{x_1+x_2}\min\left\{\binom{x_2+y_1}{x_2},\binom{x_1+y_2}{x_1}\right\}\leq R.$$

At last we are ready to prove the promised multinomial inequality. Since

$$\binom{x_1+x_2+y_1+y_2}{x_1,\ x_2,\ y_1+y_2}=\binom{x_1+x_2+y_1+y_2}{x_1+x_2}\binom{x_1+x_2}{x_1}$$

and

$$\binom{x_1+x_2+y_1+y_2}{x_1+x_2,\ y_1,\ y_2}=\binom{x_1+x_2+y_1+y_2}{x_1+x_2}\binom{y_1+y_2}{y_1}$$

it is equivalent to prove the following theorem.

Theorem 3. *For nonnegative integers* $x_1,\ x_2,\ y_1,$ *and* $y_2,$

$$\binom{x_1+x_2+y_1+y_2}{x_1+x_2}\min\left\{\binom{x_1+x_2}{x_1},\binom{y_1+y_2}{y_1}\right\}\leq R.$$

Proof. First, we interchange the roles of x_1 and y_1 in Corollary 2 to produce the correct binomial coefficients in the argument of the minimum. This gives us

$$\binom{x_1+x_2+y_1+y_2}{y_1+x_2}\min\left\{\binom{x_2+x_1}{x_2},\binom{y_1+y_2}{y_1}\right\}\leq \binom{x_1+y_1}{x_1}\binom{y_1+y_2}{y_1}\binom{x_2+x_1}{x_2}\binom{x_2+y_2}{x_2} \tag{1}$$

Observing that

$$\begin{aligned}\binom{x_1+x_2+y_1+y_2}{y_1+x_2}\binom{x_2+y_1}{x_2}\binom{x_1+y_2}{x_1} &= \frac{(x_1+x_2+y_1+y_2)!}{x_1!x_2!y_1!y_2!}\\ &= \binom{x_1+x_2+y_1+y_2}{x_1+x_2}\binom{x_1+x_2}{x_1}\binom{y_1+y_2}{y_1},\end{aligned}$$

and then multiplying both sides of (1) by $\dfrac{\binom{x_2+y_1}{x_2}\binom{x_1+y_2}{x_1}}{\binom{x_1+x_2}{x_1}\binom{y_1+y_2}{y_1}}$ will yield the desired result.

We note that equality holds in Theorem 3 if and only if two of $x_1,\ x_2,\ y_1,$ and y_2 are zero. □

This inequality arose during an investigation into *strong* edge colorings of bipartite graphs. An edge coloring is strong if the distance between two edges of the same color is at least 2. In [1], it was conjectured that a strong edge coloring of any bipartite graph, $G = (X, E, Y)$, requires at most $\Delta(X)\Delta(Y)$ colors, where $\Delta(S)$ denotes the maximum degree of the vertices in S. For a restricted class of bipartite graphs investigated in [2], the conjectured $\Delta(X)\Delta(Y)$ bound equals R, the right-hand side in Theorem 3. Two strong edge colorings are constructed using $\binom{x_1 + x_2 + y_1 + y_2}{x_1,\ x_2,\ y_1 + y_2}$ and $\binom{x_1 + x_2 + y_1 + y_2}{x_1 + x_2,\ y_1,\ y_2}$ colors respectively. This note establishes that at least one of these colorings obeys the $\Delta(X)\Delta(Y)$ upper bound.

References

[1] R.A. Brualdi and J.J. Quinn, Incidence and strong edge colorings of graphs, *Discrete Math.* **122** (1993) 51–58.

[2] J.J. Quinn and A.T. Benjamin, Strong chromatic index of subset graphs, *J. Graph Theory*, Vol. 24 No. 3, (1997) 267–273.

Department of Mathematics, Harvey Mudd College, 1250 Dartmouth Avenue, Claremont, CA 91711

E-mail address: `benjamin@hmc.edu`

Department of Mathematics, Occidental College, 1600 Campus Drive, Los Angeles, CA 90041.

E-mail address: `jquinn@oxy.edu`

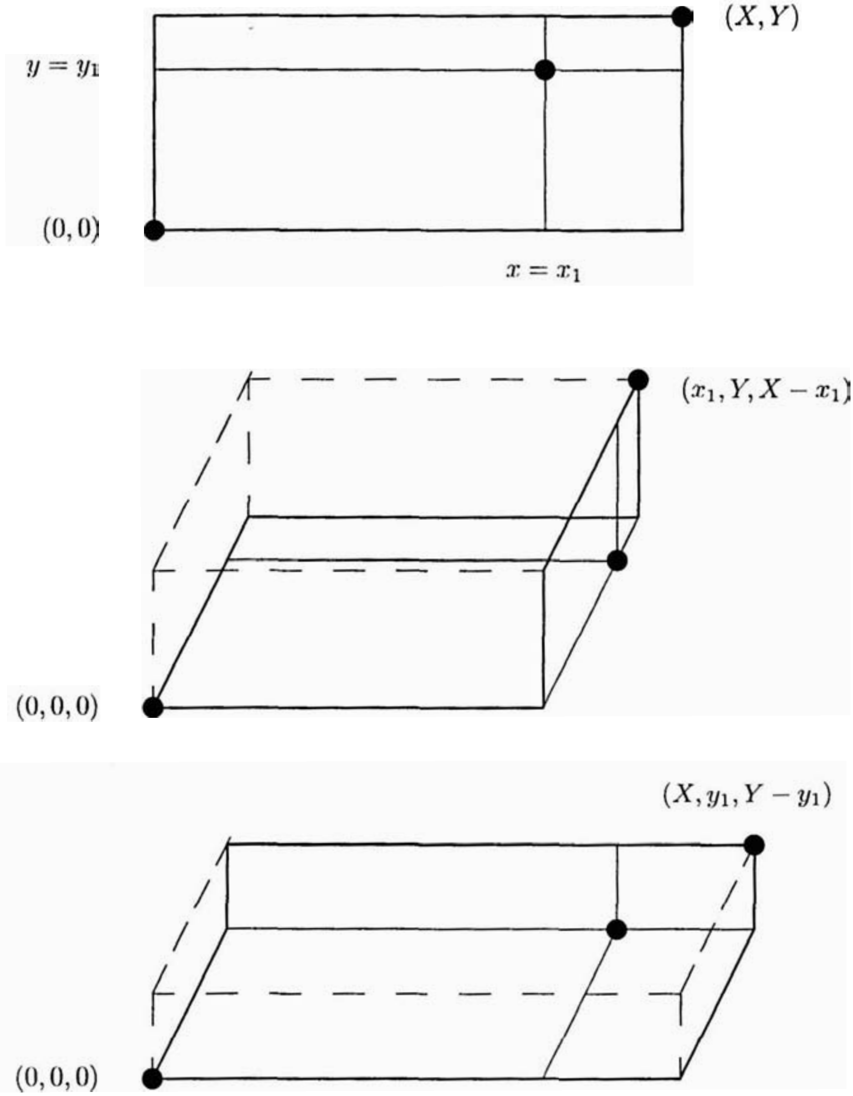

FIGURE 1. Folding a lattice induces two boxes. We show that at least one of the above boxes has fewer increasing antipodal paths than the number of ways to select increasing paths from the four quadrants created by the folds.

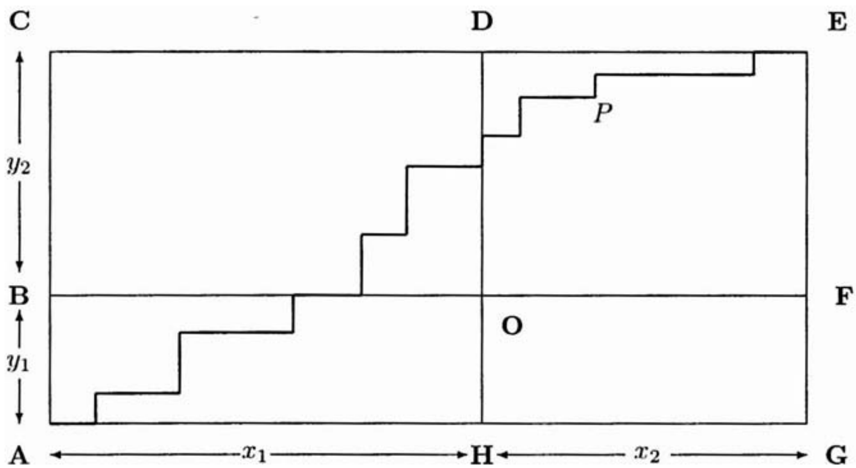

FIGURE 2. An increasing path from A to E.

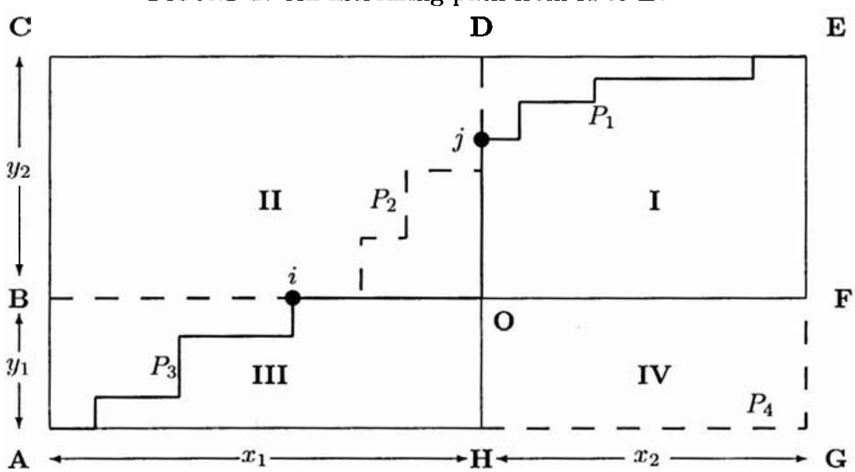

FIGURE 3. Corresponding increasing paths for each quadrant.