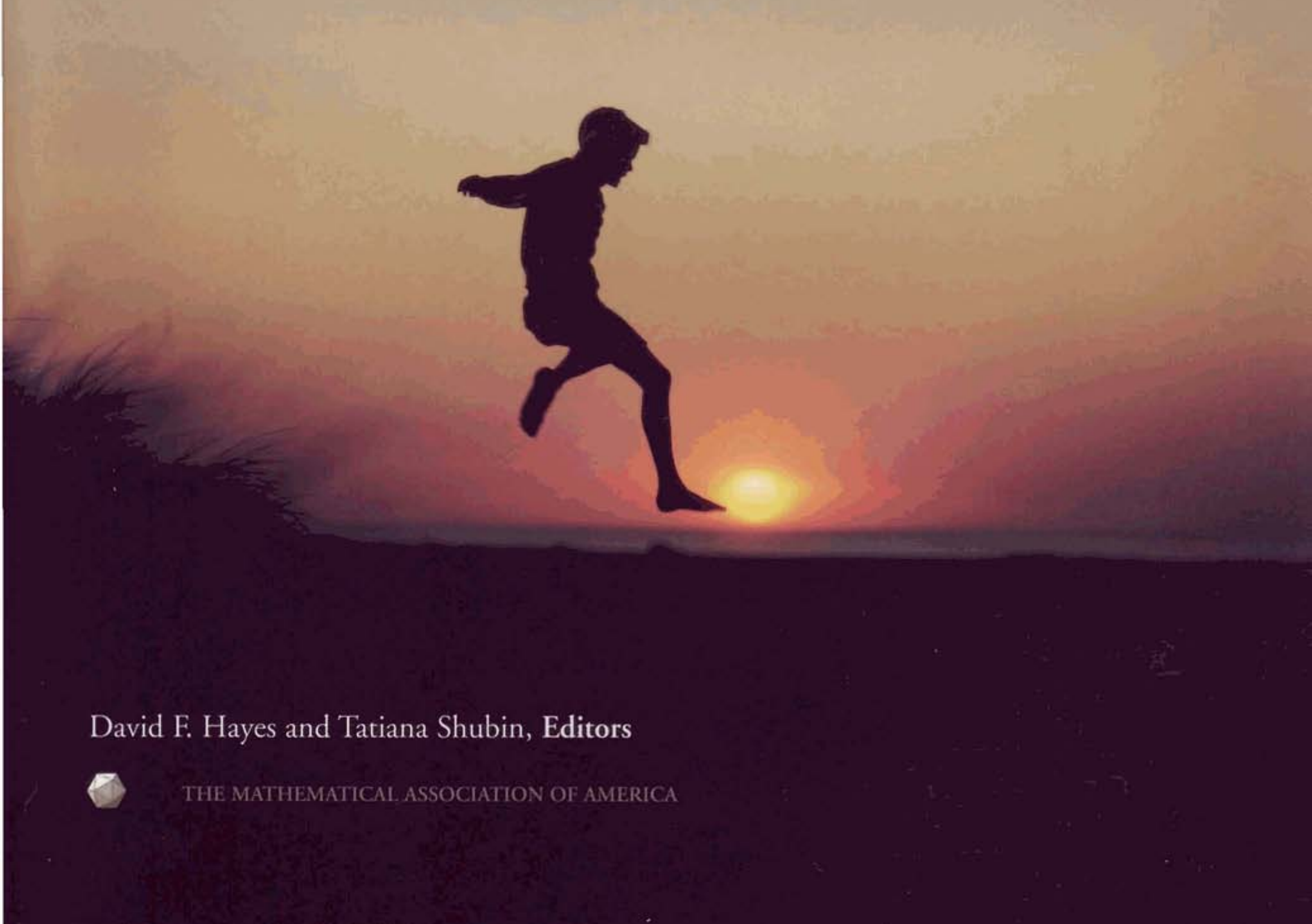
Mathematical Adventures for Students and Amateurs



David F. Hayes and Tatiana Shubin, **Editors**

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Mathematical Adventures for Students and Amateurs

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9

Proofs that Really Count: The Magic of Fibonacci Numbers and More

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A Magic Trick

A mathemagician hands a sheet of paper as in Figure 9.1 to a volunteer and says, "Secretly write a positive integer in Row 1 and another positive integer in Row 2. Next, add those numbers together and put the sum in Row 3. Add Row 2 to Row 3 and place the answer in Row 4. Continue in this fashion until numbers are in Rows 1 through 10. Now, using a calculator if you wish, add all the numbers in Rows 1 through 10 together." While the spectator is adding, the mathemagician glances at the sheet of paper for just a second, then instantly reveals the total. "Now using a calculator, divide the number in Row 10 by the number in Row 9, and announce the first three digits of your

1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
TOTAL	

Figure 9.1. Enter positive integers in Rows 1 and 2. The number in each successive row is the sum of the numbers in the previous two rows.

1	x
2	y
3	$x+y$
4	$x+2y$
5	$2x+3y$
6	$3x+5y$
7	$\mathbf{5x+8y}$
8	$8x+13y$
9	$13x+21y$
10	$21x+34y$
TOTAL	$\mathbf{55x+88y}$

Figure 9.2. The sum of the 10 numbers is Row 7 times 11.

answer. What's that you say? 1.61? Now turn over the paper and look what I have written." The back of the paper says "I predict the number 1.61".

A direct explanation of this trick involves nothing more than high school algebra. For the first part, observe in Figure 9.2 that if Row 1 contains x and Row 2 contains y then the total of Rows 1 through 10 will sum to $55x + 88y$. As luck (or is it something more?) would have it, the number in Row 7 is $5x + 8y$. Consequently, the grand total is simply 11 times Row 7, and with practice, even large numbers can be mentally multiplied by eleven.

As for the ratio, it's all about adding fractions badly. For any two fractions $\frac{a}{b} < \frac{c}{d}$ with positive numerators and denominators, the quantity $\frac{a+c}{b+d}$ is called the *mediant* (sometimes called the *freshman sum*) and it's easy to show that

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

Consequently, the ratio of (Row 10)/(Row 9) satisfies

$$1.615\ldots = \frac{21}{13} = \frac{21x}{13x} < \frac{21x+34y}{13x+21y} < \frac{34y}{21y} = \frac{34}{21} = 1.619\ldots$$

This magic trick is an application of some special properties of the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,..., where each number is the sum of the previous two. Their many beautiful patterns are a constant source of amazement. For instance, the magic trick above is assisted by the fact that the sum of the first n Fibonacci numbers is one less than the $(n+2)$nd Fibonacci number. Here, we reveal interpretations of Fibonacci numbers and related sequences to demystify their secrets—requiring nothing more than the ability to count.

Fibonacci Numbers

How many sequences of 1's and 2's sum to n? Let's call the answer to this counting question f_n. For example, $f_4 = 5$ since 4 can be created in the following 5 ways:

$$1+1+1+1, \quad 1+1+2, \quad 1+2+1, \quad 2+1+1, \quad 2+2.$$

1	2	3	4	5	6
1	11	111	1111	11111	111111
	2	12	112	1112	11112
		21	121	1121	11121
			211	1211	11211
			22	122	1122
				2111	12111
				212	1212
				221	1221
					21111
					2112
					2121
					2211
					222
$f_1 = 1$	$f_2 = 2$	$f_3 = 3$	$f_4 = 5$	$f_5 = 8$	$f_6 = 13$

Table 9.1. f_n and the sequence of 1's and 2's summing to n for $n = 1, 2, \ldots, 6$.

Table 9.1 illustrates the values of f_n for small n. The pattern is unmistakable; f_n begins like the Fibonacci numbers. In fact, it will continue to grow like Fibonacci numbers, that is for $n > 2$, f_n satisfies

$$f_n = f_{n-1} + f_{n-2}.$$

To see this, we consider the first number in our sequence. If the first number is 1, the rest of the sequence sums to $n - 1$, so there are f_{n-1} ways to complete the sequence. If the first number is 2, there are f_{n-2} ways to complete the sequence. Hence, $f_n = f_{n-1} + f_{n-2}$.

For our purposes, we prefer a more visual representation of f_n. By considering the 1's as representing *squares* and the 2's as representing *dominoes*, f_n counts the number of ways to *tile* a board of length n with squares and dominoes. For simplicity, we call a length n board an *n-board.* Thus $f_4 = 5$ enumerates the tilings given in Figure 9.3.

Figure 9.3. All five square-domino tilings of the 4-board.

We let $f_0 = 1$ count the empty tiling of the 0-board. Thus for $n \geq 0$, we have a combinatorial interpretation of the nth Fibonacci number:

f_n counts the number of ways to tile a length n board with squares and dominoes.

This interpretation allows many Fibonacci identities to be proved by asking a counting question and answering it in two different ways. Since both expressions are answers to the same question, they must be equal. For example, the sum of consecutive Fibonacci numbers can be explained as follows:

Identity 1 $f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$

Question: How many tilings of an $(n + 2)$-board use at least one domino?

Answer 1: There are f_{n+2} tilings of an $(n+2)$-board. Excluding the "all square" tiling gives $f_{n+2} - 1$ tilings with at least one domino.

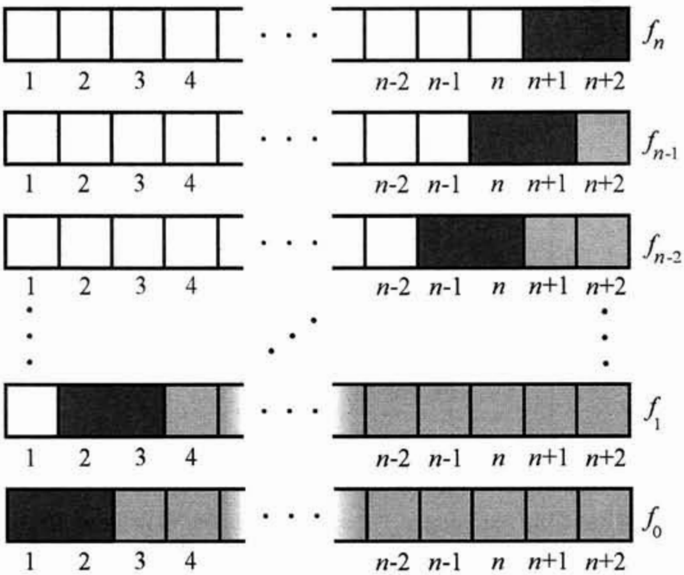

Figure 9.4. To see that $f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$, tile an $(n+2)$-board with squares and dominoes and consider the location of the last domino.

Answer 2: Consider the location of the last domino. There are f_k tilings where the last domino covers cells $k+1$ and $k+2$. This is because cells 1 through k can be tiled in f_k ways, cells $k+1$ and $k+2$ must be covered by a domino, and cells $k+3$ through $n+2$ must be covered by squares. Hence the total number of tilings with at least one domino is $f_0 + f_1 + f_2 + \cdots + f_n$. See Figure 9.4.

Since our logic is impeccable in both answers, they must be equal and the identity follows.

Many Fibonacci identities depend on the notion of breakability at a given cell. We say that a tiling of an n-board is *breakable* at cell k, if the tiling can be broken into two tilings, one covering cells 1 through k and the other covering cells $k+1$ through n. On the other hand, we call a tiling *unbreakable* at cell k if a domino occupies cells k and $k+1$. For example, the tiling of the 10-board in Figure 9.5 is breakable at cells $1, 2, 3, 5, 7, 8, 10$, and unbreakable at cells $4, 6, 9$. Notice that the tiling of an n-board (henceforth abbreviated an *n-tiling*) is always breakable at cell n. We apply these ideas to the next identity.

Identity 2 $f_{m+n} = f_m f_n + f_{m-1} f_{n-1}$.

Question: How many tilings of an $(m+n)$-board exist?

Answer 1: There are f_{m+n} $(m+n)$-tilings.

Answer 2: Consider breakability at cell m.

An $(m+n)$-tiling that is breakable at cell m is created from an m-tiling followed by an n-tiling. There are $f_m f_n$ of these.

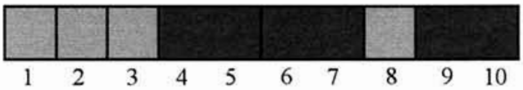

Figure 9.5. A 10-tiling that is breakable at cells 1, 2, 3, 5, 7, 8, 10 and unbreakable at cells 4, 6, 9.

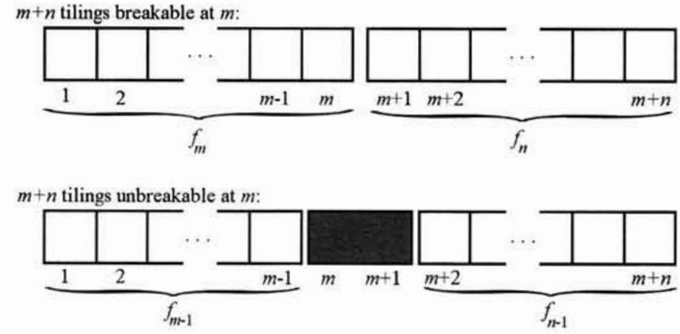

Figure 9.6. To prove $f_{m+n} = f_m f_n + f_{m-1} f_{n-1}$ count $(m+n)$-tilings based on whether or not they are breakable or unbreakable at m.

An $(m+n)$-tiling that is unbreakable at cell m must contain a domino covering cells m and $m+1$. So the tiling is created from an $(m-1)$-tiling followed by a domino followed by an $(n-1)$-tiling. There are $f_{m-1} f_{n-1}$ of these.

Since a tiling is either breakable or unbreakable at cell m, there are $f_m f_n + f_{m-1} f_{n-1}$ tilings altogether. See Figure 9.6.

Another combinatorial proof technique is to interpret both sides of an identity as sizes of two different sets and then find a one-to-one correspondence between them. We apply this idea to the next identity and we introduce the useful technique of *tail swapping*.

Consider the two 10-tilings offset as in Figure 9.7. The first one tiles cells 1 through 10; the second one tiles cells 2 through 11. We say that there is a *fault* at cell i, $2 \le i \le 10$, if both tilings are breakable at cell i. We say there is a fault at cell 1 if the first tiling is breakable at cell 1. Put another way, the pair of tilings has a fault at cell i, $1 \le i \le 10$, if neither tiling has a domino covering cells i and $i+1$. The pair of tilings in Figure 9.7 has faults at cells 1, 2, 5, and 7. We define the *tail* of a tiling to be the tiles that occur after its last fault. Observe that if we swap the tails of Figure 9.7 we obtain the 11-tiling and the 9-tiling in Figure 9.8, and it has the same faults.

Looking at Identity 3, it may appear that the $(-1)^n$ term prevents us from proving it combina-

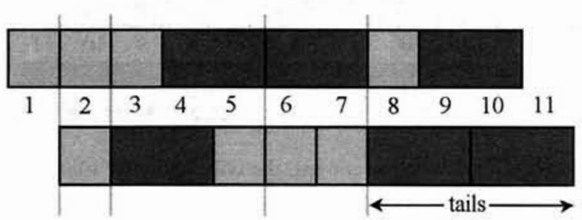

Figure 9.7. Two 10-tilings with their faults (indicated with gray lines) and tails.

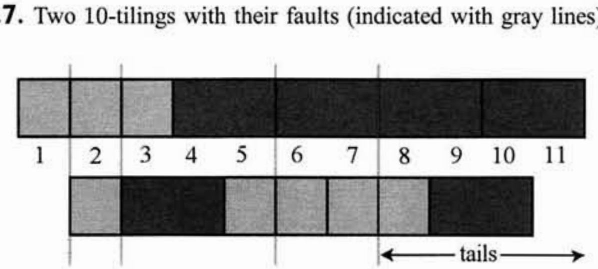

Figure 9.8. After tail-swapping, we have an 11-tiling and a 9-tiling with exactly the same faults.

torially. Nonetheless, we will see that this term is merely the error term of an "almost" one-to-one correspondence.

Identity 3 $f_n^2 = f_{n+1}f_{n-1} + (-1)^n$

Set 1: Tilings of two n-boards (a *top* board and a *bottom* board.) By definition, this set has size f_n^2.

Set 2: Tilings of an $(n+1)$-board and an $(n-1)$-board. This set has size $f_{n+1}f_{n-1}$.

Correspondence: First, suppose n is odd. Then the top and bottom board must each have at least one square. Notice that a square in cell i ensures that a fault must occur at cell i or cell $i-1$. Swapping the tails of the two n-tilings produces an $(n+1)$-tiling and an $(n-1)$-tiling with the same tails. This produces a 1-to-1 correspondence between all pairs of n-tilings and all tiling pairs of sizes $n+1$ and $n-1$ that have faults. Is it possible for a tiling pair of sizes $n+1$ and $n-1$ to be "fault free"? Yes, when all the dominoes are in "staggered formation" as in Figure 9.9. Thus, when n is odd, $f_n^2 = f_{n+1}f_{n-1} - 1$.

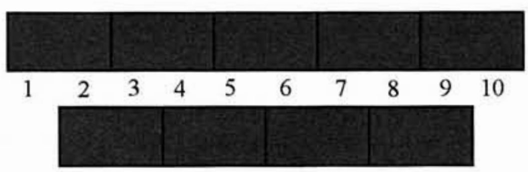

Figure 9.9. When n is odd, the only fault-free tiling pair.

Similarly, when n is even, tail swapping creates a 1-to-1 correspondence between faulty tiling pairs. The only fault-free tiling pair is the all domino tiling of Figure 9.10. Hence, $f_n^2 = f_{n+1}f_{n-1} + 1$. Considering the odd and even cases together produces our identity.

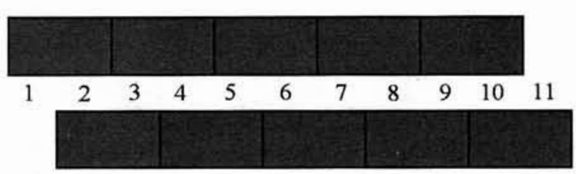

Figure 9.10. When n is even, the only fault-free tiling pair.

We invite readers to try their hand at combinatorially proving the Fibonacci identities below. Recall that $\binom{n}{k}$ counts the number of ways to select k elements from an n element set. For $0 \le k \le n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. When $n < 0$, $k < 0$, or $k > n$, we have $\binom{n}{k} = 0$.

$$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n} - 1.$$

$$f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1}.$$

$$f_{n+2} + f_{n-2} = 3f_n.$$

$$\sum_{k=0}^{n} \binom{n-k}{k} = f_n.$$

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n-i}{j}\binom{n-j}{i} = f_{2n+1}.$$

$$f_n + f_{n-1} + f_{n-2} + 2f_{n-3} + 4f_{n-4} + 8f_{n-5} + \cdots + 2^{n-2}f_0 = 2^n.$$

Lucas Numbers

Close companions to the Fibonacci numbers, are the Lucas numbers 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123,..., where each term is the sum of the previous two terms but the initial conditions are different. As we shall see Lucas numbers operate like Fibonacci numbers running in circles.

Let us combinatorially define L_n to be the number of ways to tile a circular board of length n with (slightly curved) squares and dominoes. For example $L_4 = 7$ as illustrated in Figure 9.11. Clearly there are more ways to tile a *circular n-board* than a straight n-board since it is now possible for a single domino to cover cells n and 1. We define an *n-bracelet* to be a tiling of a circular n-board. A bracelet is *out-of-phase* when a single domino covers cells n and 1 and *in-phase* otherwise. In Figure 9.11, we see that there are 5 in-phase 4-bracelets and 2 out-of-phase 4-bracelets. Figure 9.12 illustrates that $L_1 = 1$, $L_2 = 3$, and $L_3 = 4$. Notice that there are two ways to create a 2-bracelet with a single domino – either in-phase or out-of-phase.

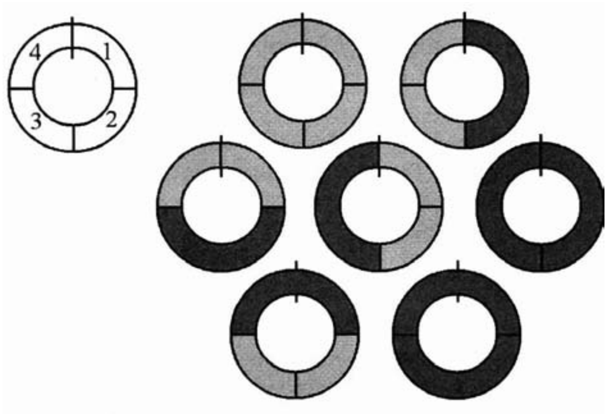

Figure 9.11. A circular 4-board and its 7 bracelets. The first 5 bracelets are in-phase and the last 2 are out-of-phase.

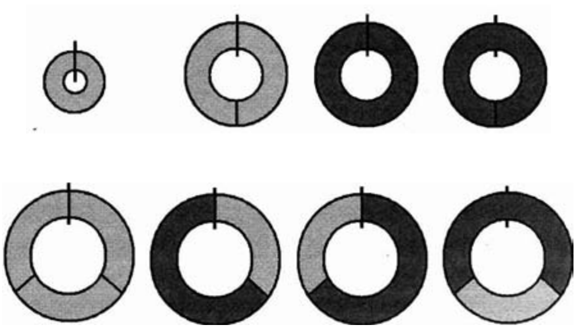

Figure 9.12. There are one 1-bracelets, three 2-bracelets, and four 3-bracelets.

From our initial data, the number of n-bracelets looks like the Lucas sequence. To prove that they continue to grow like the Lucas sequence, we must argue that for $n \geq 3$,

$$L_n = L_{n-1} + L_{n-2}.$$

To see this we simply consider the *last tile* of the bracelet. We define the *first* tile to be the tile that covers cell 1, which could either be a square, a domino covering cells 1 and 2, or a domino covering cells n and 1. The second tile is the next tile in the clockwise direction, and so on. The last tile is the one that precedes the first tile. Since it is the first tile, not the last, that determines the phase of the tiling, there are L_{n-1} n-bracelets that end with a square and L_{n-2} n-bracelets that end with a domino. By removing the last tile, we produce smaller bracelets.

To make the recurrence valid for $n = 2$, we define $L_0 = 2$, and interpret this to mean that there are two empty tilings of the circular 0-board, an in-phase 0-bracelet and an out-of-phase 0-bracelet. Thus for $n \geq 0$, we have a combinatorial interpretation of the nth Lucas number:

L_n counts the number of ways to tile a circular board of length n with squares and dominoes.

As one might expect, there are many identities with Lucas numbers that resemble Fibonacci identities. In addition, there are many beautiful identities where Lucas and Fibonacci numbers interact.

Identity 4 $L_n = f_n + f_{n-2}$.

Question: How many tilings of a circular n-board exist?

Answer 1: By definition, there are L_n n-bracelets.

Answer 2: Consider whether the tiling is in-phase or out-of-phase. Since an in-phase tiling can be straightened into an n-tiling, there are f_n in-phase bracelets. Likewise, an out-of-phase n-bracelet must have a single domino covering cells n and 1. Cells 2 through $n-1$ can then be covered as a straight $(n-2)$-tiling in f_{n-2} ways. Hence the total number of n-bracelets is $f_n + f_{n-2}$. See Figure 9.13.

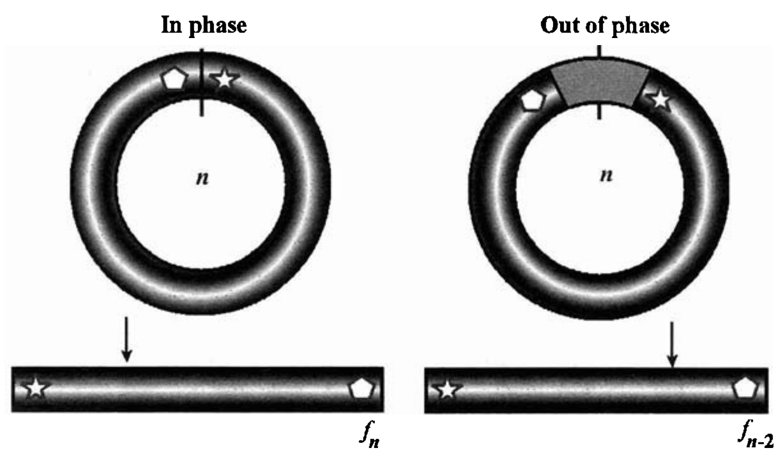

Figure 9.13. Every circular n-bracelet can be reduced to an n-tiling or an $(n-2)$-tiling, depending on its phase.

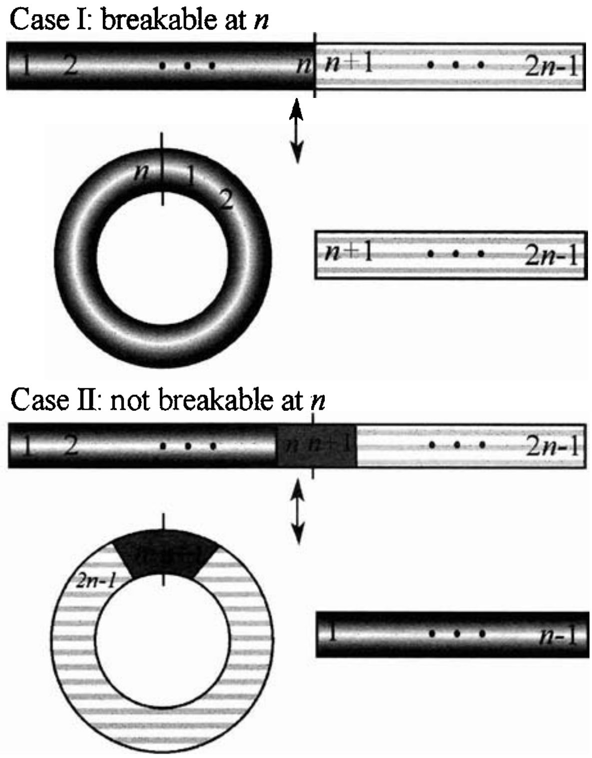

Figure 9.14. A $(2n-1)$-tiling can be converted to an n-bracelet and $(n-1)$-tiling. In our correspondence, the n-bracelet is in-phase if and only if the $(2n-1)$-tiling is breakable at cell n.

Identity 5 $f_{2n-1} = L_n f_{n-1}$.

Set 1: Tilings of a $(2n-1)$-board. This set has size f_{2n-1}.

Set 2: Bracelet-tiling pairs (B, T) where the bracelet has length n and the tiling has length $n-1$. This set has size $L_n f_{n-1}$.

Correspondence: Given a $(2n-1)$-board T^*, there are 2 cases to consider, as illustrated in Figure 9.14.

Case I: If T^* is breakable at cell n, then glue the right side of cell n to the left side of cell 1 to create an in-phase n-bracelet B, and cells $n+1$ through $2n-1$ form an $(n-1)$-tiling T.

Case II: If T^* is unbreakable at cell n, then cells n and $n+1$ are covered by a domino which we denote by d. Cells 1 through $n-1$ become an $(n-1)$-tiling T and cells n through $2n-1$ are used to create an out-of-phase n-bracelet with d as its first tile.

This correspondence is easily reversed since the phase of the n-bracelet indicates whether Case I or Case II is invoked. So the correspondence is a bijection and Set 1 and Set 2 have the same size.

The reader may wish to prove these Lucas identities combinatorially.

$$L_{m+n} = f_m L_n + f_{m-1} L_{n-1}.$$

$$5f_n = L_n + L_{n+2}.$$
$$L_n^2 = L_{n+1}L_{n-1} + (-1)^n \cdot 5.$$
$$L_0 + 2L_1 + 4L_2 + 8L_3 + \cdots + 2^n L_n = 2^{n+1} f_n.$$
$$\sum_{k=0}^{n} 5^k \binom{n}{2k} = 2^{n-1} L_n.$$

Gibonacci Numbers

Gibonacci number is shorthand for generalized Fibonacci number. We say a sequence of nonnegative integers $G_0, G_1, G_2, \ldots$ is a *Gibonacci sequence* if for all $n \geq 2$,

$$G_n = G_{n-1} + G_{n-2}.$$

Such sequences are completely determined by G_0 and G_1. For instance, the Lucas sequence is the Gibonacci sequence beginning with $G_0 = 2$ and $G_1 = 1$. To see how to interpret these numbers combinatorially, we take a second look at Lucas numbers. From the previous section we know that L_n counts the number of ways to tile an n-bracelet with squares and dominoes. Notice that we can "straighten out" an n-bracelet, by writing it as an n-tiling starting with the first tile (the tile covering cell 1) with one caveat. The caveat is that if the first tile is a domino, we need to indicate whether it is an in-phase or out-of-phase domino. For example, the seven 4-bracelets of Figure 9.11 have been straightened out in *phased tilings* in Figure 9.15. In summary, L_n counts the number of *phased* n-tilings where an initial domino has **2** possible phases and an initial square has **1** possible phase. The next theorem should then come as no surprise.

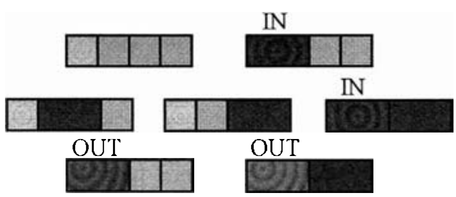

Figure 9.15. The seven 4-bracelets can be straightened out to become "phased" 4-tilings.

Theorem Let $G_0, G_1, G_2, \ldots$ be a Gibonacci sequence with nonnegative integer terms. For $n \geq 1$, G_n counts the number of n-tilings, where the initial tile is assigned a phase. There are G_0 choices for a domino phase, and G_1 choices for a square phase.

Proof. Let a_n denote the number of phased n-tilings with G_0 and G_1 phases for initial dominoes and squares, respectively. Clearly, $a_1 = G_1$. A phased 2-tiling consists of either a phased domino (G_0 choices) or a phased square followed by an unphased square (G_1 choices). Hence $a_2 = G_0 + G_1 = G_2$. To see that a_n grows like Gibonacci numbers we consider the last tile, which immediately gives us $a_n = a_{n-1} + a_{n-2}$. □

In order for our theorem to be valid when $n = 0$, we combinatorially define the number of phased 0-tilings to be G_0, the number of domino phases. Using this combinatorial interpretation of G_n, we observe that many identities become transparent. For instance, from the shape of the first tile of a phased tiling (see Figure 9.16), it immediately follows that

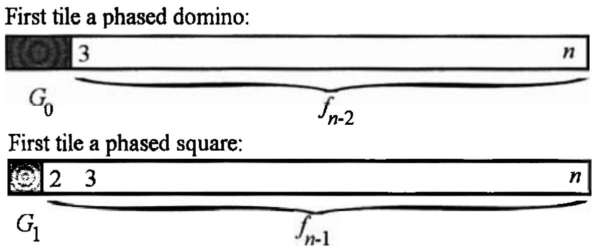

Figure 9.16. A phased n-tiling either begins with a phased domino or a phased square.

Identity 6 $G_n = G_0 f_{n-2} + G_1 f_{n-1}$

The next two identities are generalizations of Identities 1 and 2 respectively.

Identity 7 $\sum_{k=0}^{n} G_k = G_{n+2} - G_1$.

Question: How many phased $(n+2)$-tilings contain at least one domino?

Answer 1: There are G_{n+2} phased $(n+2)$-tilings including the G_1 tilings consisting of only squares. So there are $G_{n+2} - G_1$ tilings with at least one domino.

Answer 2: Consider the location of the last domino. For $0 \le k \le n$, there are G_k tilings where the last domino covers cells $k+1$ and $k+2$ as illustrated in Figure 9.17. Notice that when the last domino covers cells 1 and 2, it must have one of G_0 phases. So the argument is still valid.

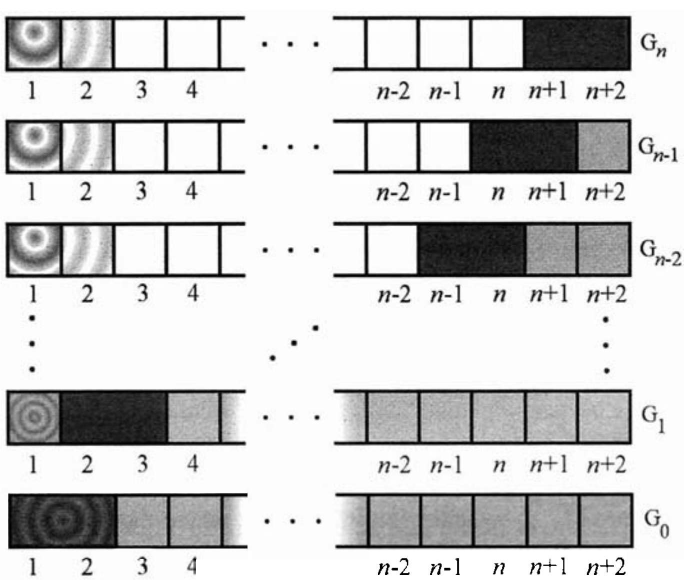

Figure 9.17. Here we consider the location of the last domino.

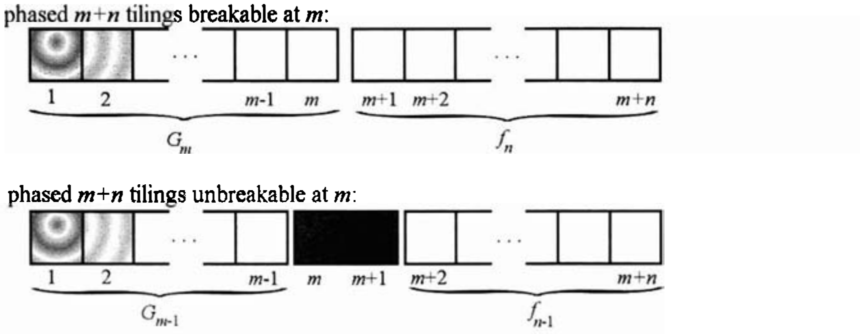

Figure 9.18. A phased $(m+n)$-tiling is either breakable or unbreakable at cell m.

Identity 8 $G_{m+n} = G_m f_n + G_{m-1} f_{n-1}$.

Question: How many phased $m+n$ tilings exist?

Answer 1: By definition, there are G_{m+n} such tilings.

Answer 2: Consider whether or not the phased $(m+n)$-tiling is breakable at cell m. See Figure 9.18. The number of breakable tilings is $G_m f_n$ since such a tiling consists of a phased m-tiling followed by a standard n-tiling. The number of unbreakable tilings is $G_{m-1} f_{n-1}$ since such tilings contain a phased $(m-1)$-tiling, followed by a domino covering cells m and $m+1$, followed by a standard $(n-1)$-tiling. Altogether, there are $G_m f_n + G_{m-1} f_{n-1}$ $(m+n)$-tilings.

The next identity uses tail-swapping on phased tilings to create an almost one-to-one correspondence with a nontrivial error term.

Identity 9 *For* $0 \le m \le n$, $G_{n+m} = G_n L_m + (-1)^{m-1} G_{n-m}$.

Set 1: The set of phased $(n+m)$-tilings. This set has size G_{n+m}.

Set 2: The set of ordered pairs (A, B), where A is a phased n-tiling, and B is an m-bracelet. This set has size $G_n L_m$.

Correspondence: We create an almost one-to-one correspondence between these two sets. Let P be a phased $(n+m)$-tiling. If P is breakable at cell n, then we create a phased n-tiling A from the phased tiling of the first n cells of P. Using cells $n+1$ through $n+m$ create B, an in-phase m-bracelet, as in Figure 9.19. If P is not breakable at cell n, then create the tiling pair of Figure 9.20, where the top tiling is the phased $(n-1)$-tiling from cells 1 through $n-1$ of P. The bottom tiling is an unphased $(m+1)$-tiling, beginning with a domino, from cells n through $n+m$ of P. Now perform a tail swap, if possible, to create a pair of tilings with sizes n and m, where the n-tiling is phased, and the m-tiling is unphased, but begins with a domino. These become a phased tiling and out-of-phase bracelet in the natural way.

When is tail-swapping not possible? When m is even, then the $(m+1)$-tiling must have at least one square, resulting in at least one fault. Thus when m is even, we can always tail-swap, but there are G_{n-m} unachievable tiling pairs where the bottom m-tiling consists of all dominoes and the phased n-tiling has only dominoes in cells $n-m+1$ through n. See Figure 9.21. Thus when m is even, $G_n L_m = G_{n+m} + G_{n-m}$ as desired. By a similar argument, when m is odd, $G_{n+m} = G_n L_m + G_{n-m}$.

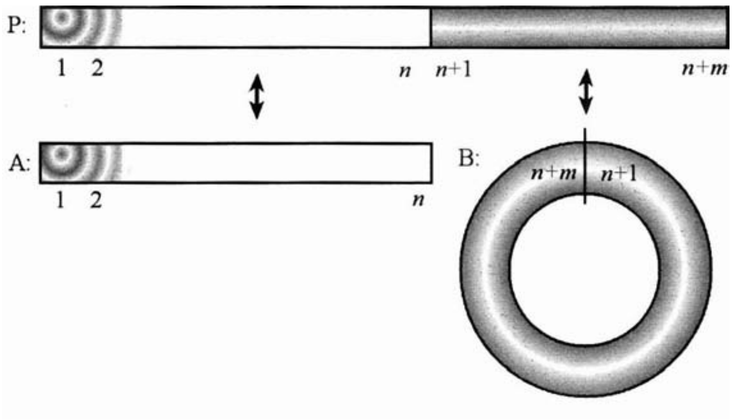

Figure 9.19. A breakable phased $(n + m)$-tiling naturally becomes a phased n-tiling with an in-phase m-bracelet.

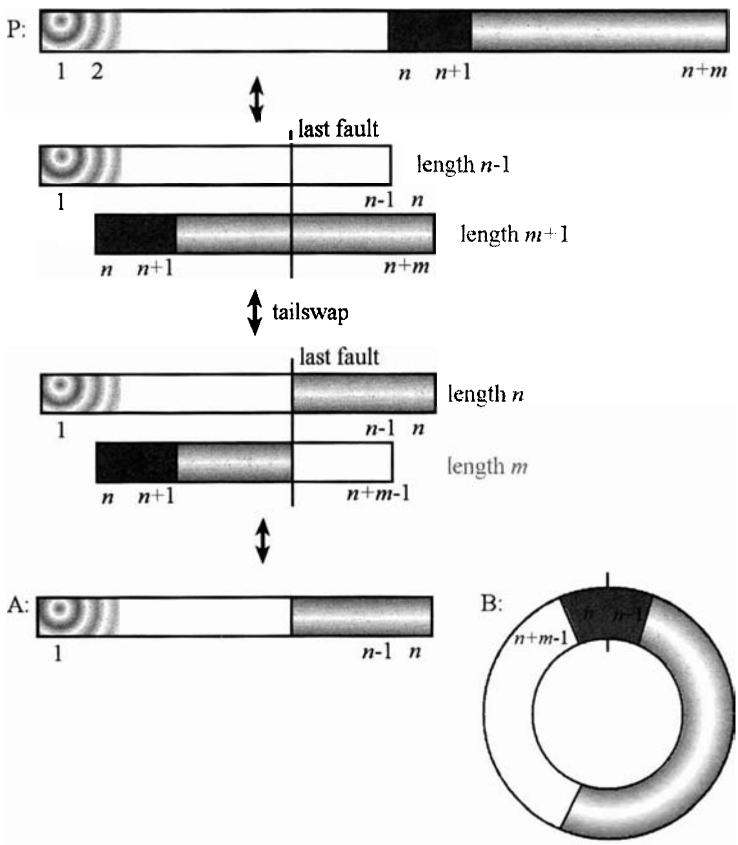

Figure 9.20. An unbreakable phased $(n + m)$-tiling becomes a phased n-tiling with an out-of-phase m-bracelet.

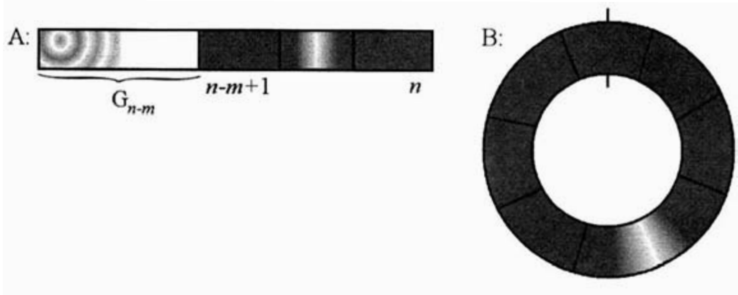

Figure 9.21. When m is even, these pairs are unachievable.

As a consequence of Identities 7 and 9, we sum the first $4n+2$ terms of any Gibonacci sequence as

$$\sum_{i=0}^{4n+1} G_i = G_{4n+3} - G_1 = G_{2n+2}L_{2n+1} + (-1)^{2n}G_1 - G_1 = G_{2n+2}L_{2n+1}$$

leading to the following identity.

Identity 10

$$\sum_{i=0}^{4n+1} G_i = L_{2n+1}G_{2n+2}.$$

Our opening magic trick, $G_0 + G_1 + \cdots + G_9 = 11 \cdot G_6$, is an application of this identity when $n = 2$. So it is no coincidence that the multiplier 11 is the fifth Lucas number.

Once more, we challenge the reader to prove combinatorially the following Gibonacci identities.

$$\sum_{k=1}^{n} G_{2k-1} = G_{2n} - G_0.$$

$$G_1 + \sum_{k=0}^{n} G_{2k} = G_{2n+1}.$$

$$\text{For } n \geq p, \quad G_{n+p} = \sum_{i=0}^{p} \binom{p}{i} G_{n-i}.$$

$$G_{m+(t+1)p} = \sum_{i=0}^{p} \binom{p}{i} f_t^i f_{t-1}^{p-i} G_{m+i}.$$

$$\sum_{i=1}^{2n} G_i G_{i-1} = G_{2n}^2 - G_0^2$$

$$\sum_{i=2}^{2n+1} G_{i-1}G_i = G_{2n+1}^2 - G_1^2.$$

$$\sum_{i=1}^{n-1} G_{i-1}G_{i+2} = G_n^2 - G_1^2.$$

$$G_{n+1}G_{n-1} - G_n^2 = (-1)^n (G_1^2 - G_0 G_2).$$

$$G_{n+1}+G_n+G_{n-1}+2G_{n-2}+4G_{n-3}+8G_{n-4}+\cdots 2^{n-1}G_0=2^n(G_0+G_1).$$

Let $G_0, G_1, G_2, \ldots$ and $H_0, H_1, H_2, \ldots$ be Gibonacci sequences. Then for $n, h, k \geq 0$,

$$G_m H_n - G_{m-1}H_{n+1} = (-1)^m[G_0 H_{n-m+2} - G_1 H_{n-m+1}].$$

Combinatorial proofs of all the previously listed identities (and more) can be found in the references at the end of this paper or in our book [5].

Open Problems

The techniques presented here are simple but powerful. Counting tilings enables us to visualize relationships between Fibonacci numbers and their generalizations. This approach facilitates a clearer understanding of existing identities and can be extended in a number of ways. By introducing colored tiles of various lengths, we can interpret sequences generated by linear recurrences with constant coefficients [1]. By allowing some of the squares of our tiling to be stacked up to a certain height, we can combinatorially interpret simple continued fractions [6]. By introducing an element of randomness, even the irrationally looking Binet formulas

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

and

$$L_n = \left[\left(\frac{1+\sqrt{5}}{2}\right)^{n} + \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right],$$

can be rationalized [1, 2].

To indicate the power of our approach, the classic book *Fibonacci & Lucas Numbers and the Golden Section* by Steven Vajda [10] contains 118 identities involving Fibonacci, Lucas, and Gibonacci numbers. These identities are proved by a myriad of algebraic methods—induction, generating functions, hyperbolic functions, to name a few. Although *none* are proved combinatorially in the book, we have used tiling to explain 91 of these identities—and counting!

We leave the reader with some of the more tantalizing identities which have thus far resisted combinatorial explanations.

$$\sum_{i=0}^{2n}\binom{2n}{i}f_{2i-1} = 5^n f_{2n-1}$$

$$\sum_{i=0}^{2n}\binom{2n}{i}f_{i-1}^2 = 5^{n-1}L_{2n}$$

$$\sum_{i=0}^{2n}\binom{2n}{i}L_{2i} = 5^n L_{2n}$$

We have every confidence that these too will be combinatorially explained someday. You can count on it.

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