Random Approaches to Fibonacci Identities

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There is no problem in all mathematics that cannot be solved by direct counting.
—Ernst Mach

1. INTRODUCTION. Many combinatorialists live by Mach's words, and take it as a personal challenge. For example, nearly all of the Fibonacci identities in [**5**] and [**6**] have been explained by counting arguments ([**1**], [**2**], [**3**]). Among the holdouts are those involving infinite sums and irrational quantities. However, by adopting a probabilistic viewpoint, many of the remaining identities can be explained combinatorially. As we demonstrate, even the irrational-looking Binet formula for the n-th Fibonacci number

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \tag{1}$$

can be rationalized by this approach.

We introduce several models for generating *random tilings* that yield fundamental identities for Fibonacci numbers and their generalizations. The models share the feature that the probability of every length n tiling depends only on n.

We begin by recalling a combinatorial interpretation of Fibonacci numbers. Let c_n denote the number of series of 1's and 2's that sum to n. Then $c_n = F_{n+1}$ since $c_1 = 1 = F_2$, $c_2 = 2 = F_3$, and (by conditioning on the first number in the series) $c_n = c_{n-1} + c_{n-2}$. Thus, for $n \geq 1$, F_n may be *combinatorially defined* as the number of ways to tile a board of length $n-1$ using squares and dominoes.

2. BLACK AND WHITE MODEL. For the first model, consider an infinite board with cells $1, 2, \ldots$ and color each cell black or white, independently, with probability $1/2$. See Figure 1 for an example. Any coloring of the first n cells has probability $(1/2)^n$.

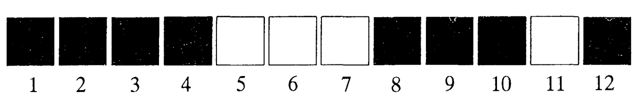

Figure 1. A random black-white board.

An infinite tiling can be viewed as alternating strings of black and white cells of varying lengths. For example, the tiling in Figure 1 has a black string of length 4 followed by a white string of length 3 followed by a black string of length 3 followed by a white string of length 1, and so on.

Let the random variable X denote the location of the end of the first black string of odd length. In our example, $X = 10$. Now we ask the question "For $n \geq 1$, what is the probability that $X = n$?" To answer this, observe that a tiling has $X = n$ if and only if cell n is covered by a black square, cell $n+1$ is covered by a white square, and cells 1 through $n-1$ can be covered by white squares and black dominoes. There are F_n ways to tile the first $n-1$ cells in this manner; therefore, $P(X=n)=F_n/2^{n+1}$. Since X is finite with probability 1, this gives a combinatorial explanation of the identity

$$\sum_{n\geq 1} F_n/2^{n+1}=1.$$

A similar strategy allows us to explain

$$\sum_{n\geq 1} nF_n/2^{n+1}=5.$$

From the preceding discussion, the left side of this identity evaluates $E(X)$, the expected value of X. To prove that $E(X)=5$, we express X as the sum of three random variables,

$$X=B+2D+R.$$

B is a geometric random variable with mean 2; it denotes the location of the first black square. D is the number of *black dominoes*, i.e., pairs of black squares, that immediately follow the first black square. Here $D+1$ is a geometric random variable with mean $4/3$. Finally, R denotes the remaining number of white and black tiles needed, possibly zero, to obtain the end of the first black string of odd length. (For the tiling in Figure 1, $B=1$, $D=1$, and $R=7$.) The colors of the two squares that cover cells $B+2D+1$ and $B+2D+2$ are, with equal probability, white-white, white-black, or black-white. In the first two cases, $R=0$, since cell $B+2D$ is the end of a black string of odd length. In the third case, cell $B+2D+1$ is the end of a black string of even length, cell $B+2D+2$ has a white square, and we are back to the drawing board. Thus by the linearity of expectation,

$$\begin{aligned} E(X) &= E(B)+2E(D)+E(R) \\ &= 2+2\left(\frac{4}{3}-1\right)+\frac{2}{3}(0)+\frac{1}{3}(2+E(X)). \end{aligned} \tag{2}$$

Solving equation (2) gives us $E(X)=5$ as desired.

The preceding indentities can also be proved using a different combinatorial model, suggested to us by Stephen Maurer. It is easy to show that F_n counts the number of binary sequences of length $n-2$ with no consecutive zeros. By considering a random infinite binary string, and letting X denote the beginning of the first double zero, one again obtains $P(X=n)=F_n/2^{n+1}$ and $E(X)=5$.

3. RANDOM APPROACHES TO BINET'S FORMULA. Binet's formula was first established by Abraham De Moivre in 1718, using generating functions. It was independently rediscovered by Jacques Binet (1843) and Gabriel Lamé (1844). For our first proof of Binet's formula, we tile an infinite board by independently placing squares and dominoes, one after another. At each decision, we use a square with probability $1/\phi$ or a domino with probability $1/\phi^2$, where $\phi=(1+\sqrt{5})/2\approx 1.618$. Conveniently, $1/\phi+1/\phi^2=1$. A random example is shown in Figure 2. In this model, the probability that a tiling begins with any particular length n sequence of squares and dominoes is $1/\phi^n$.

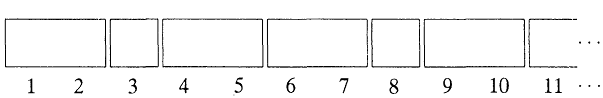

Figure 2. A random tiling of squares and dominoes.

We apply this model to derive Binet's formula. Since $(1-\sqrt{5})/2 = -1/\phi$, Binet's formula (1) says

$$F_n = \frac{1}{\sqrt{5}}\left[\phi^n - (-1/\phi)^n\right]. \tag{3}$$

Let q_n be the probability that a random tiling is *breakable* at cell n, i.e., that a square or domino begins at cell n. The example in Figure 2 is breakable at cells $1, 3, 4, 6, 8, 9, 11$, and so on. Since there are F_n different ways to tile the first $n-1$ cells,

$$q_n = F_n/\phi^{n-1}. \tag{4}$$

For a tiling to be unbreakable at n, it must be breakable at $n-1$ followed by a domino. Thus for $n \geq 2$,

$$q_n = 1 - q_{n-1}/\phi^2, \tag{5}$$

where $q_1 = F_1 = 1$. Let $q = \lim_{n\to\infty} q_n$. Taking a limit in (5) gives $q = 1 - q/\phi^2$. Solving for q we find that $q = (1 + 1/\phi^2)^{-1} = \phi/\sqrt{5}$.

Combined with (4), this gives us the asymptotic form of Binet's formula:

$$F_n \approx \phi^n/\sqrt{5}.$$

To derive Binet's formula exactly, simply unravel recurrence (5) along with initial condition $q_1 = 1$ to get

$$q_n = 1 - 1/\phi^2 + 1/\phi^4 - 1/\phi^6 + \cdots + \left(-1/\phi^2\right)^{n-1}. \tag{6}$$

This is a finite geometric series that simplifies to

$$q_n = \frac{\phi}{\sqrt{5}}\left[1 - \left(\frac{-1}{\phi^2}\right)^n\right]. \tag{7}$$

Multiplying both sides of (7) by ϕ^{n-1} yields the desired identity.

Notice that by substituting (4) into (5) and multiplying by ϕ^n we have also demonstrated

$$\phi^n = \phi F_n + F_{n-1}.$$

We briefly mention three other random approaches to Binet's formula; for a discussion of more traditional approaches, see [**4**]. The process of randomly placing squares and dominoes by the previous model can be thought of as a Markov chain that moves between breakable (B) and unbreakable (U) states according to the matrix of transition probabilities

$$P = \begin{array}{c} \\ B \\ U \end{array} \begin{array}{c} \begin{array}{cc} B & U \end{array} \\ \begin{bmatrix} 1/\phi & 1/\phi^2 \\ 1 & 0 \end{bmatrix} \end{array}$$

where p_{ij} is the probability of going from state i to state j. We begin at time (cell) 1 in the breakable state. Thus q_n, the probability that we are breakable at time n, is the $(1, 1)$ entry of P^{n-1}. By diagonalization

$$P^{n-1} = \begin{bmatrix} 1 & 1 \\ 1 & -\phi^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-1/\phi^2)^{n-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -\phi^2 \end{bmatrix}^{-1}$$

whose $(1, 1)$ entry simplifies to (7).

Another approach is to create an "almost one-to-one" correspondence between tilings breakable at n and those breakable at $n - 1$. Suppose T is a tiling, breakable at n, where n is odd. We describe a procedure for transforming T into a tiling that is breakable at $n - 1$. Define k to be the smallest number in $\{2, \dots, n\}$ such that T is breakable at cells k and $n + k - 2$. After swapping the tiles covering cells 1 through $k - 1$ with those covering cells n through $n + k - 3$ (as illustrated in Figure 3), $T = \mathbf{a}_1\mathbf{a}_2\mathbf{b}_1\mathbf{b}_2$ is transformed to $T' = \mathbf{b}_1\mathbf{a}_2\mathbf{a}_1\mathbf{b}_2$, which is breakable at $n - 1$. (When $k = 2$ this shifts the tiles covering cells 2 through $n - 1$ to the left one cell, and moves the square on cell 1 to cell $n - 1$.) The procedure, when defined, is one-to-one, onto, and "preserves probability" since the string $\mathbf{b}_1\mathbf{a}_2\mathbf{a}_1$ has the same length as $\mathbf{a}_1\mathbf{a}_2\mathbf{b}_1$. The only strings for which the procedure is undefined are those that begin with $n - 1$ dominoes, which has probability $1/\phi^{2(n-1)}$. Thus $q_n = q_{n-1} + 1/\phi^{2(n-1)}$. By a similar argument, one can show that when n is even, $q_n = q_{n-1} - 1/\phi^{2(n-1)}$. Combining the even and odd cases gives us

$$q_n = q_{n-1} + \left(-1/\phi^2\right)^{n-1},$$

which leads directly to (6).

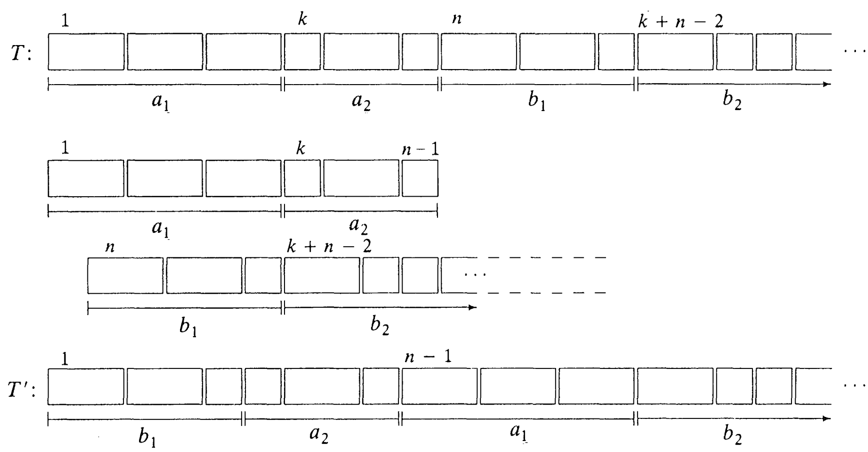

Figure 3. An almost one to one correspondence between tilings breakable at n and tilings breakable at $n - 1$.

For an approach that produces (6) without the use of a recurrence, we use a different random tiling model. Initially label the cells of an infinite board with 1's and 2's, independently, with probability $1/\phi$ and $1/\phi^2$, respectively. From this labelling, create a tiling by reading from left to right, and replacing 1 with a square, and 21 or 22 with a domino. Thus the labelling 21121221212... generates (not uniquely) the tiling of Figure 2. As in the previous model, every tiling of length n has probability $1/\phi^n$. Notice that a tiling is unbreakable at n if and only if cell $n - 1$ is labeled with 2 and the number (possibly zero) of 2's that immediately precede it is even. For example, when $n = 6$ the probability that the tiling is unbreakable at cell 6 is

$$1 - q_6 = 1/\phi^2 - 1/\phi^4 + 1/\phi^6 - 1/\phi^8 + 1/\phi^{10}.$$

The first two terms compute the probability that our initial 5 labels are of the form $abc12$, where a, b, and c are arbitrary. The next two terms compute the probability of starting with $a1222$, where a is arbitrary, and the last term is the probability of beginning with 22222. Applying the same argument to an arbitrary n yields (6) directly.

4. BEYOND BINET. The combinatorial approach can be used to prove other Fibonacci-like identities. For example, the Lucas numbers $2, 1, 3, 4, 7, 11, 18, 29, 47, \ldots$ satisfy:

$$L_n = \phi^n + (-1/\phi)^n. \tag{8}$$

As described in [**1**], L_n counts the number of ways to tile a length n board with squares and dominoes, with the caveat that if the tiling begins with a domino, the domino is assigned one of two different *phases*. In our random model, the first tile is either a square with probability $1/\sqrt{5}$, an in-phase domino with probability $1/\phi\sqrt{5}$ or an out-of-phase domino with probability $1/\phi\sqrt{5}$. Thereafter, tiles are chosen at random and independently with probability $1/\phi$ for squares and $1/\phi^2$ for dominoes. In this model, any length n tiling has probability $1/\phi^{n-1}\sqrt{5}$. Let r_n denote the probability that a tiling is breakable at $n+1$. Thus for $n \geq 2$, $r_n = L_n/\phi^{n-1}\sqrt{5}$, where $r_1 = 1/\sqrt{5}$. By the same argument as in (5), for $n \geq 2$, $r_n = 1 - r_{n-1}/\phi^2$, which unravels to

$$r_n = 1 - 1/\phi^2 + 1/\phi^4 - 1/\phi^6 + \cdots + (-1/\phi^2)^{n-2} + (-1/\phi^2)^{n-1}/\sqrt{5}.$$

Summing the series results in

$$r_n = \frac{\phi}{\sqrt{5}}\left[1 + \left(\frac{-1}{\phi^2}\right)^n\right],$$

which is the same as identity (8) with both sides divided by $\phi^{n-1}\sqrt{5}$.

Similarly, any *generalized Fibonacci* sequence $G_0, G_1, G_2, \ldots,$ satisfying $G_n = G_{n-1} + G_{n-2}$ for $n \geq 2$, can be shown to have closed form

$$G_n = \alpha\phi^n + \beta(-1/\phi)^n, \tag{9}$$

where $\alpha = (G_1 + G_0\phi)/\sqrt{5}$ and $\beta = (\phi G_0 - G_1)/\sqrt{5}$; see [**5**] or [**6**].

When G_0 and G_1 are non-negative integers, G_n can be given the combinatorial interpretation of the number of ways to tile a length n board with squares and dominoes, where there are G_0 phases for an initial domino, and G_1 phases for an initial square [**2**].

In this random model, the first tile is either a square in one of G_1 phases with probability $\phi/(G_1\phi + G_0)$ or a domino in one of G_0 phases with probability $1/(G_1\phi + G_0)$. Thereafter, tiles are chosen at random and independently with probability $1/\phi$ for squares and $1/\phi^2$ for dominoes. Here, any length n tiling has probability $1/(G_1\phi + G_0)\phi^{n-2}$. Then for $n \geq 2$, unravelling the probability that a tiling is breakable at n allows (9) to be explained combinatorially.

REFERENCES

1. A. T. Benjamin and J. J. Quinn, Recounting Fibonacci and Lucas Identities, *College Math. J.* 30 (1999) 359–366.
2. A. T. Benjamin, J. J. Quinn, and F. E. Su, Generalized Fibonacci Identities through Phased Tilings, *Fibonacci Quart.*, to appear.

3. R. C. Brigham, R. M. Caron, P. Z. Chinn, and R. P. Grimaldi, A Tiling Scheme for the Fibonacci Numbers, *J. Recreational Math.*, 28 (1996–97) 10–16.
4. R. J. Hendel, Approaches to the Formula for the nth Fibonacci Number, *College Math. J.* 25 (1994) 139–142.
5. V. E. Hoggatt, Jr., *Fibonacci and Lucas Numbers*, The Fibonacci Association, Santa Clara, CA, 1979.
6. S. Vajda, *Fibonacci Lucas Numbers, and the Golden Section: Theory and Applications*, John Wiley and Sons, New York, 1989.

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