# READER reflections

## DIVISION BY 1001

One of the authors of this note has publicly prognosticated ("Ta Da!" *New York Times*, Education Life section, December 30, 2009) that if one divides a whole number by 7 (and does not get an integer result), then the sum of the first six digits after the decimal point will be 27. Of course, this must be true—the six digits after the decimal point will always be 1, 4, 2, 8, 5, and 7, although possibly in a different order.

For a teacher workshop on fractions and decimals, a modified problem was given:

1. Choose a three-digit number.
2. Feel lucky to be here? Divide by 7.
3. Feel a little less lucky after seeing that result? Divide by 11.
4. Feel unlucky after seeing that result? Divide by 13.
5. Now add the sum of the first six digits after the decimal point.

When the trick is performed this way, the surprising result is that the sum of the first six digits after the decimal point is still 27, but the digits do not follow a pattern as simple as the one that occurs when dividing by 7. The mathematics behind why this trick works, however, is quite beautiful.

Successively dividing by 7, 11, and 13 is equivalent to dividing by 1001. Notice what happens when a three-digit number is divided by 1001:

$$\frac{159}{1001} = 0.\overline{158841}$$

The first three digits after the decimal point are 1 less than the numerator, and the last three digits are the difference between the numerator and 1000. In addition, if the six digits in the repeating cycle are separated into two three-digit numbers, their sum is 158 + 841 = 999. More generally, when a three-digit number of the form $100a + 10b + c$ is divided by 1001, the result is as follows:

$$\begin{aligned}\frac{100a+10b+c}{1001} &= \frac{999(100a+10b+c)}{999 \cdot 1001}\\ &= \frac{1000\left(100a+10b+c-1\right)}{999{,}999}\\ &\quad + \frac{1000}{999{,}999} - \frac{\left(100a+10b+c\right)}{999{,}999}\end{aligned}$$

The numerator will be a six-digit number. The denominator of 999,999 indicates that these six digits will occur in a repeating cycle when the fraction is expressed as a decimal. Moreover, the first three digits will be $n - 1$; the last three digits will be $1000 - n$; and their sum is $(n - 1) + (1000 - n) = 999$. Consequently, the sum of the digits will always be 9 + 9 + 9 = 27.

Of course, this result can be generalized. If a $k$-digit number $n$ is divided by $10^k + 1$, the result is a repeating decimal whose cycle takes the form $10^k(n - 1) + (10^k - n)$, and the sum of the first $2k$ digits after the decimal point is $9k$. For instance, if $n = 123{,}456$, then $k = 6$, $n - 1 = 123{,}455$, $10^k - n = 876{,}544$, and $9k = 54$. Consequently,

$$\frac{123{,}456}{1{,}000{,}001} = 0.\overline{123455876544},$$

and 1 + 2 + 3 + 4 + 5 + 5 + 8 + 7 + 6 + 5 + 4 + 4 = 54.

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[Look for problems based on this idea in future *MT* Calendars.—*Ed.*]

## COMPLETING THE SQUARE: A NEW METHOD

Algebra is a course that all students experience, albeit some with less-than-enthusiastic attitudes. In the many opportunities I have had to teach algebra, one topic that always elicits groans from students has been completing the square.

What is so gut wrenching about this method that makes students want to skip the problem altogether? What is it about this concept that makes it so difficult? Why would any student prefer to memorize that long quadratic formula instead of simply adding a quantity here and there?

I suspected that the stumbling block was factoring. It never ceases to amaze me how many of my students in any given semester have trouble factoring any expression, much less a quadratic.

To get around the factoring issue, I worked backward from the goal of the completed-square form. Starting with what some call the vertex form of a quadratic equation, I multiplied and simplified all the quantities. Here the vertex of the parabola is $(h, k)$:

$$a\left(x-h\right)^2 + k = 0$$
$$a\left(x^2 - 2hx + h^2\right) + k = 0$$
$$ax^2 - 2ahx + ah^2 + k = 0$$

Now that I had a quadratic equation in general form, I could match up these

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quantities with the ones given in the problem and solve for $h$ and $k$. Following is an example:

Solve $3x^2 - 5x + 1 = 0$.

We compare this equation to the model $ax^2 - 2ahx + ah^2 + k = 0$ and set $a = 3$. We can then set $-2ah = -5$ and substitute to obtain $h = 5/6$. Using these values for $a$ and $h$ and substituting into $ah^2 + k = 1$, we see that $k = -13/12$.

We now know $a$, $h$, and $k$, so we can use them in the standard form of the quadratic equation and solve for $x$:

$$3x^2 - 5x + 1 = 0 \rightarrow$$
$$3\left(x - \frac{5}{6}\right)^2 - \frac{13}{12} = 0$$
$$3\left(x - \frac{5}{6}\right)^2 = \frac{13}{12}$$
$$\left(x - \frac{5}{6}\right)^2 = \frac{13}{36}$$
$$x - \frac{5}{6} = \pm\frac{\sqrt{13}}{6}$$
$$x = \frac{5}{6} \pm \frac{\sqrt{13}}{6}$$

This new method also works when dealing with circles. Suppose that an example gives the equation of a circle in the general form $x^2 + ax + y^2 + by + c = 0$ and that the task is to find the center $(h, k)$ and the radius $r$. First, look at the standard form of the equation of a circle: $(x - h)^2 + (y - k)^2 = r^2$. Multiply and simplify the quantities as shown:

$$(x - h)^2 + (y - k)^2 = r^2$$
$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$$
$$x^2 - 2hx + y^2 - 2ky + h^2 + k^2 - r^2 = 0$$

Now compare with the general form, setting $a = -2h$, $b = -2k$, and $c = h^2 + k^2 - r^2$.

Although this method may have more steps, the typical pitfalls in solving this type of problem are removed. Notice that we solve for $h$, $k$, and $r$ directly. In the completing-the-square method, students must remember that when they see $(x - 3)^2$, $h$ is 3, not $-3$.

Every student should be held responsible for understanding factoring procedures. However, it may be possible to make the process of finding an answer more straightforward by using basic algebra techniques. Perhaps this method will lead to a better understanding of finding perfect square trinomials and of completing the square.

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## OF THE INTERSECTION OF TWO ARTICLES

Reading both "A Lesson on the Slopes of Perpendicular Lines," by John M. Tucker, and "Translations toward Connected Mathematics," by Mark Applebaum and Roza Leikin (*Mathematics Teacher* April 2010, vol. 103, no. 8, pp. 603–7 and 562–69, respectively) led me to remember a simple, clear proof of this theorem that uses transformations and that is intuitive and very convincing. Twenty-five years ago I was thrilled to replace the old two-column proof (which usually required three congruent triangles and some twenty steps!) with the following. The key is comparing the original line with its rotated copy.

In the coordinate plane, choose any two points on any line $l_1$. For simplicity's sake, let the slope be positive, although a negative slope will also work. Draw the right triangle as shown in **figure 1 (Kalman)**, following the grid lines. Tracing the legs from left to right, represent the legs as $a$ and $b$. Then both $a$ and $b$ are positive, and the slope of the line is $b/a$.

Now rotate the line and the triangle 90° about any point on $l_1$. The new line, $l_2$, is perpendicular to the original line $l_1$ and has its own right triangle. Consider the legs of this triangle on $l_2$. The vertical leg on $l_1$ has become the horizontal leg on $l_2$, and the horizontal leg on $l_1$ has become the vertical leg on $l_2$. Further, as we view the line from left to right, the vertical leg drops, traveling in a negative direction. Denote its length as $-a$. Thus, the slope of $l_2$ is $-a/b$, which is the negative reciprocal of the slope of $l_1$.

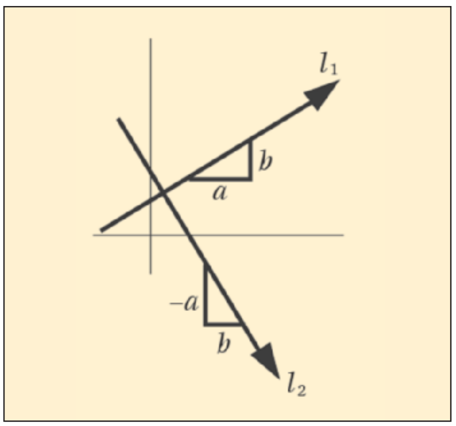


**Fig. 1 (Kalman)**

The proof of the converse is quite similar and just as intuitive. My students had no trouble accepting its truth immediately.

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## CORRECTION: HISTORY OF TEACHING TRIGONOMETRY

David M. Bressoud's article "Historical Reflections on Teaching Trigonometry" (*MT* September 2010, vol. 104, no. 2, pp. 106–12) intrigued me. I have taught a one-semester high school trigonometry course for approximately fifteen years. In it, I first teach right-triangle ratio trigonometry (for angles measured in degrees) for about a month and then circle trig (for distances measured in a generic unit of measure called radians) for another month. Of course, the relationships that evolve from the unit circle and that connect those two concepts are heavily stressed. The rest of the course involves both types of problems, and the student must determine how to proceed. However, I have never before seen the history of the development of trigonometry, and I was intrigued. In the future, I intend to include lessons on both the fundamental problem of trigonometry (as described in the article) and the history involved.

I am sure, however, that I am not the first to mention the typographical error in figure 5. Since the angle $\theta$ is formed by the diameter shown and the radius marked $R$, the bracket indicating the measure of the half-chord should be labeled $R\sin\theta$.

Thank you for the very interesting article.

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