Recounting the Sums of Cubes of Fibonacci Numbers

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Arthur T. Benjamin

Timothy A. Carnes

Benoit Cloitre

Dept. of Mathematics, Harvey Mudd College, Claremont, CA 91711

benjamin@hmc.edu, tcarnes@hmc.edu, abcloitre@modulonet.fr

In the book *Proofs that Really Count* [1], the authors prove over 100 Fibonacci identities by combinatorial arguments, but they leave some identities unproved and invite the readers to find combinatorial proofs of these. The first uncounted identity concerns the sum of the cubes of Fibonacci numbers.

Identity 1. *For* $n \geq 0$*,*

$$\sum_{k=0}^{n} f_k^3 = \frac{f_{3n+4} + (-1)^n 6 f_{n-1} + 5}{10},$$

where $f_n = F_{n+1}$.

In this paper, we present a simple combinatorial proof of this identity, and then use the same tools to prove an even simpler closed form.

It is well-known that f_n counts the ways to tile a one-dimensional board of length n using squares of length one and dominoes of length two. We refer to such tilings as n-tilings, and we let $\mathcal{F}_n$ denote the set of all n-tilings. An n-tiling, which covers cells 1 through n, is said to be *breakable* at cell k if cells k and $k+1$ are not covered by a domino. For example, the 9-tiling in Figure 1 is breakable at cells $2, 3, 5, 7, 8, 9$, and can be described using the notation $dsddss$ or dsd^2s^2, where s denotes a square and d denotes a domino.

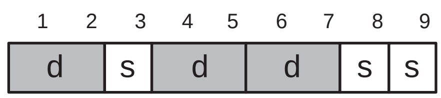

Figure 1: A 9-tiling that is breakable at cells $2, 3, 5, 7, 8, 9$.

The following identity will be useful to us and it serves as an easy warm-up exercise.

Identity 2. *For $n \geq 0$,*

$$2\sum_{k=0}^{n} f_{3k+2} = f_{3n+4} - 1.$$

Proof. There is just one $(3n+4)$-tiling that is not breakable at any of the cells of the form $3j+2$, namely $s(ds)^{n+1}$ consisting of a square followed by dominoes and squares in alternating fashion. Therefore $f_{3n+4} - 1$ counts all of the $(3n+4)$-tilings that are breakable at at least one cell of the form $3j+2$. The left side of the identity counts the same problem by considering the last breakable cell of the form $3j+2$. If this occurs at cell $3k+2$, then cells 1 through $3k+2$ can be tiled in f_{3k+2} ways, and cells $3k+3$ and $3k+4$ can be tiled two ways (either with two squares or with a single domino). Thereafter, to avoid breakable cells of the form $3j+2$, there is just one way to continue, namely $(ds)^{n-k}$, consisting of dominoes and squares in alternating fashion. See Figure 2. Altogether, as k ranges from 0 to n we have $2\sum_{k=0}^{n} f_{3k+2}$ tilings. Since the same set has been counted two different ways, their sizes must be equal. □

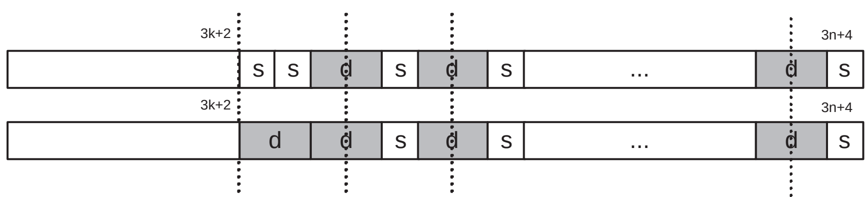

Figure 2: Using a $(3k+2)$-tiling to create a $(3n+4)$-tiling whose last breakable cell of the form $3j+2$ is $3k+2$.

Let $T = \cup_{k=0}^{n} \mathcal{F}_{3k+2}$ denote the set of tilings of length $3k+2$ for some k satisfying $0 \leq k \leq n$. Thus T has size $\sum_{k=0}^{n} f_{3k+2}$.

Let U denote the set of $(3n+4)$-tilings excluding the tiling $s(ds)^{n+1}$, so U has size $f_{3n+4} - 1$. The proof of Identity 2 establishes a one-to-two correspondence between T and U. Next we define S to be the set of triples of tilings (A, B, C) where A, B, C all belong to $\mathcal{F}_k$ for some $0 \leq k \leq n$. Clearly S has size $\sum_{k=0}^{n} f_k^3$. To prove Identity 1, which we rewrite as

$$10 \sum_{k=0}^{n} f_k^3 = f_{3n+4} + (-1)^n 6 f_{n-1} + 5,$$

we establish an *almost* one-to-ten correspondence between S and U, where the error term of $(-1)^n 6 f_{n-1} + 6$ will be explained later. This will be accomplished by an almost one-to-five correspondence between S and T, followed by the previously described one-to-two correspondence between T and U.

To understand the mapping from S to T, we present two other identities, whose combinatorial proofs will be of use to us. First is the familiar Cassini Identity whose combinatorial proof, presented in [1], is given here to establish the notation.

Identity 3. *For $k \geq 1$,*

$$f_k^2 - f_{k+1} f_{k-1} = (-1)^k.$$

Proof. We establish an almost one-to-one correspondence between $\mathcal{F}_k \times \mathcal{F}_k$ and $\mathcal{F}_{k+1} \times \mathcal{F}_{k-1}$, by a technique known as *tailswapping*. Given an ordered pair of k-tilings (X, Y) where X has cells 1 through k and Y covers cells 2 through $k+1$, we construct the tiling pair (X^+, Y^-) by swapping the tiles of X and Y after the last cell for which X and Y are both breakable. See Figure 3. The resulting tiling pair (X^+, Y^-) belongs to $\mathcal{F}_{k+1} \times \mathcal{F}_{k-1}$ since the tiles of X^+ cover cells 1 through $k+1$, while the tiles of Y^- cover cells 2 through k. Naturally, this procedure can be easily reversed by applying the same rule. The only tiling pairs (X, Y) or (X^+, Y^-) that can not be tailswapped are those tilings consisting of all dominoes. When k is even, this can only occur when $(X, Y) = (d^{k/2}, d^{k/2})$; whence, $f_k^2 - f_{k+1}f_{k-1} = 1$. When k is odd, the only unswappable tiling pair is $(X^+, Y^-) = (d^{(k+1)/2}, d^{(k-1)/2})$; whence, $f_k^2 - f_{k+1}f_{k-1} = -1$. □

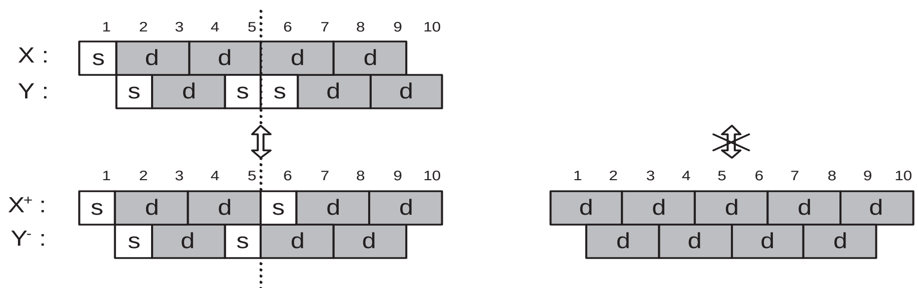

Figure 3: A pair of 9-tilings can always be tailswapped to obtain a 10-tiling and an 8-tiling. In the first example, both tiling pairs are breakable at cells 1 and 5. The procedure is easily reversed, except when the 10-tiling and 8-tiling both consist entirely of dominoes.

Finally, we define for a nonempty k-tiling X, the *conjugate* of X to be the same as X but with the last tile switched from a square to a domino or from a domino to a square. For example, the conjugate of the 9-tiling *dsddss* is the 10-tiling *dsddsd*. We denote the conjugate by $\overline{X}$ which has length $\overline{k}$ equal to $k+1$ or $k-1$ depending on whether X ended with a square or domino, respectively. It is clear that $\overline{\overline{X}} = X$ and that k and $\overline{k}$ are always of opposite parity. The following identity appears in [1] as Identity 21. Using conjugation leads to an even simpler combinatorial proof than the one presented there.

Identity 4. *For $n \geq 1$,*

$$\sum_{k=1}^{n} (-1)^k f_k = (-1)^n f_{n-1}.$$

Proof. Consider the set of non-empty tilings of length at most n. Pairing up each tiling with its conjugate establishes an almost one-to-one correspondence between the tilings of even length and the tilings of odd length. There are exactly f_{n-1} tilings in this set that do not have a conjugate in the set, namely those n-tilings that end with a square (since their conjugates would have length $n+1$). Consequently, in this set, the number of even-length tilings minus the number of odd-length tilings is $(-1)^n f_{n-1}$, as desired. □

We are now in a position to prove Identity 1.

Proof. First we consider the case where n is odd, where we show that

$$10 \sum_{k=0}^{n} f_k^3 - 6 + 6 f_{n-1} = f_{3n+4} - 1.$$

Here we will describe a mapping that sends every non-empty triple (A, B, C) in S to exactly five tilings in T, that in turn become ten tilings in U. The empty triple $(\emptyset, \emptyset, \emptyset)$ is sent to just two elements of T (namely the 2-tilings s^2

and d) that in turn become four tilings of U, instead of ten. (This accounts for the -6 term.) Also, we will show that the mapping from S to T fails to hit exactly $3f_{n-1}$ elements of T, which generate $6f_{n-1}$ additional elements of U. Except for triples (A, B, C) where two or more of A, B, C consist entirely of dominoes, the triple $(A, B, C) \in \mathcal{F}_k^3$ will be mapped to five tilings of length $3k + 2$ as follows, where a superscript of $+$ or $-$ refer respectively to gaining or losing a cell through tailswapping.See also Figure 4.

1. $AsBsC$. (Translation: Insert a square between A and B and a square between B and C.)

2. $ABdC$.

3. A^-dB^+C, unless $(A, B, C) = (d^{k/2}, d^{k/2}, C)$ for some even k.

4. A^+BdC^-, unless $(A, B, C) = (d^{k/2}, B, d^{k/2})$ for some even k.

5. AdB^-C^+, unless $(A, B, C) = (A, d^{k/2}, d^{k/2})$ for some even k.

Before explaining how to handle the triples where two or more of A, B, C consist entirely of dominoes, we first observe that this mapping is easily reversed by observing the contents of cells $k + 1$ and $2k + 2$. The contents of any cell is restricted to three possibilities, namely a square, a left-end of a domino, or a right-end of a domino, which we will respectively refer to as S, L, and R. Since there are only three possibilities and we are considering two cells, then there are a total of nine possible outcomes. By inspecting Figure

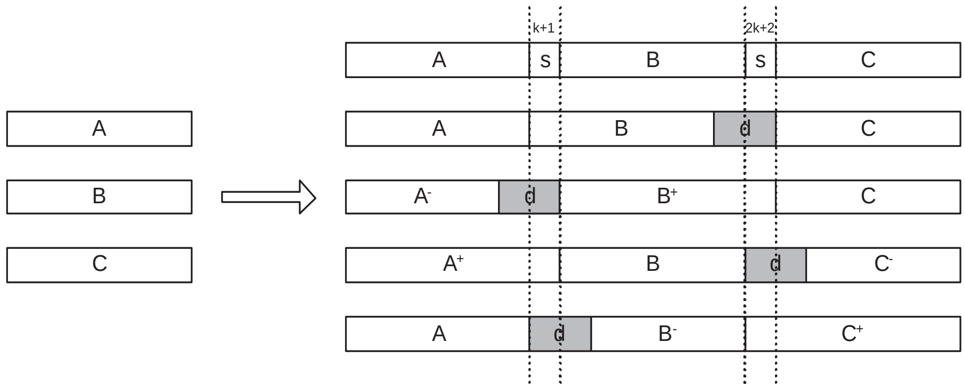

Figure 4: A mapping from a triple of k-tilings to five tilings of length $(3k+2)$.

4, we see that the first $(3k+2)$-tiling will necessarily contain a square in both cells $k+1$ and $2k+2$, which we denote by SS. The second $(3k+2)$-tiling will have R in cell $2k+2$, but it could have either S or L in cell $k+1$. Hence the second $(3k+2)$-tiling must be of the form SR or LR. The other three tilings are demonstrated in the following table.

Tiling	Cells $k+1$ and $2k+2$
1	SS
2	SR or LR
3	RS or RR
4	SL or RL
5	LS or LL

In our mapping, tiling 1 will hit every element of T denoted by SS and tiling 2 will hit every element of T denoted by SR or LR. However, tilings 3, 4, and 5 will miss some elements of T because of tailswapping. Specifically, all elements of T that correspond to tiling 3 are covered except for those where A^- and B^+ consist entirely of dominoes (which forces k to be odd).

Thus, tiling 3 fails to hit those elements of T of the form $d^{k+1}C$, where C has odd length k. Likewise, tilings 4 and 5 respectively miss those tilings in T of the form $d^{(k+1)/2}Bd^{(k+1)/2}$ and Ad^{k+1} where B and A have odd length k.

This suggests a natural way to handle tiling 3 for the cases not covered by the original mapping. When $(A,B,C) = (d^{k/2}, d^{k/2}, C)$ for some even k, then we replace C with its conjugate $\overline{C}$ with odd length $\overline{k}$, and map it to the $(3\overline{k}+2)$-tiling $d^{\overline{k}+1}\overline{C}$. Notice that since n is odd, and k is even, we must have $k \leq n-1$ and therefore $\overline{k} \leq n$, so this mapping is well-defined for all (A,B,C) considered here. Likewise, we can extend tilings 4 and 5 in a similar way. For tiling 4, if $(A,B,C) = (d^{k/2}, B, d^{k/2})$ for some even k, then replace B with its conjugate $\overline{B}$ with odd length $\overline{k}$ and map it to $d^{(\overline{k}+1)/2}\overline{B}d^{(\overline{k}+1)/2}$. For tiling 5, if $(A,B,C) = (A, d^{k/2}, d^{k/2})$ for some even k, then map it to $\overline{A}d^{\overline{k}+1}$.

Extending tilings 3, 4, and 5 as just described ensures that every element of S (except for the empty triple) is mapped to exactly five elements of T. As predicted, there are precisely $3f_{n-1}$ elements of T that are unhit by this mapping. These unhit elements of T are of the form $d^{n+1}\overline{C}$, $d^{(n+1)/2}\overline{B}d^{(n+1)/2}$, or $\overline{A}d^{n+1}$, where $\overline{A}, \overline{B}, \overline{C}$ have odd length n and end in a square. These elements of T are unhit because the conjugates of $\overline{A}, \overline{B}, \overline{C}$ would each have length $n+1$ and therefore not belong to S.

When n is even, we apply the same mappings that we used when n is odd. In this situation, every element of T is hit by this mapping. On the other hand, the mapping is undefined for exactly $3f_{n-1}$ elements of S, namely those (A, B, C) of the form $(d^{n/2}, d^{n/2}, C)$ or $(d^{n/2}, B, d^{n/2})$ or $(A, d^{n/2}, d^{n/2})$, where A, B, C are n-tilings that end in a square. The mapping is undefined here because the conjugates of A, B, and C would have length $n+1$ causing the images of these tilings to have length $3n+5$, and hence not belong to T. Consequently, these $3f_{n-1}$ elements of S will only be defined for four of the five cases,and thereby only generate eight elements of U instead of ten. Thus, when n is even, we have

$$10 \sum_{k=0}^{n} f_k^3 - 6 - 6f_{n-1} = f_{3n+4} - 1,$$

as desired. □

Next we present a new identity for the sum of cubes of Fibonacci numbers. The identity is simpler than the first one by having a smaller denominator, which leads to a simpler combinatorial proof.

Identity 5. *For $n \geq 0$,*

$$\sum_{k=0}^{n} f_k^3 = \frac{f_n f_{n+1}^2 + (-1)^n f_{n-1} + 1}{2}.$$

Proof. We begin by answering a simple counting question in two different ways.

QUESTION: How many ways can one create an ordered triple (A, B, C), where A is a tiling of length n, B is a tiling of length $n+1$ and C is a tiling of length n+1?

ANSWER 1: By tiling A, B, and C separately, there are $f_n f_{n+1}^2$ such triples.

ANSWER 2: Next, we claim that the number of such triples is also $\sum_{k=0}^{n}(f_k^3 + f_{k-1}f_k f_{k+1})$. Suppose that A covers cells 1 through n, B covers cells 1 through $n+1$, and C covers cells 1 through $n+1$. For $0 \leq k \leq n$, we shall define (A, B, C) to have *parameter* k if k is the largest cell for which either (i) A, B, and C are all breakable at cell k or (ii) C has a domino covering cells k and $k+1$. Notice that conditions (i) and (ii) are mutually exclusive since if C has a domino covering cells k and $k+1$ then it is not breakable at cell k.

We point out that the case $k = 0$ only corresponds to the triple (A, B, C) where C consists entirely of squares, $(C = s^{n+1})$, and A and B are "fault free". That is, when n is even, A consists of all dominoes $(A = d^{n/2})$ and B begins with a square followed by all dominoes $(B = sd^{n/2})$. When n is odd, $A = sd^{(n-1)/2}$ and $B = d^{(n+1)/2}$.

The number of tilings with parameter k that satisfy condition (i) is f_k^3, since cells 1 through k of A, B, and C can each be tiled f_k ways. The rest of the tiling is forced so as to prevent k from being larger: tiling C must contain all squares on cells $k+1$ through $n+1$ and all cells bigger than k of

A and B must be tiled fault free in the unique way.

The number of tilings that satisfy condition (ii) with parameter k is $f_{k-1}f_kf_{k+1}$. For this to happen, tiling C can be created f_{k-1} ways (arbitrary tiling of cells 1 through $k-1$, followed by a domino on cells k and $k+1$, followed by all squares on cells $k+1$ through $n+1$). In order for A and B to be fault free after cell k, one of them (depending on the parity of $n-k$) must contain all dominoes from cell $k+1$ to the end, and the other one must have all dominoes from cell $k+2$ to the end. Thus A and B can be tiled in f_kf_{k+1} ways. Altogether, condition (ii) occurs $f_{k-1}f_kf_{k+1}$ ways.

Summing over all possible values of the parameter k, we have that Answer 2 equals $\sum_{k=0}^{n}(f_k^3 + f_{k-1}f_kf_{k+1})$.

Equating Answers 1 and 2, gives us the identity

$$f_nf_{n+1}^2 = \sum_{k=0}^{n}(f_k^3 + f_{k-1}f_kf_{k+1}). \qquad (1)$$

If we allow ourselves the algebraic luxury of Identities 3 and 4, then we have

$$\begin{aligned} f_nf_{n+1}^2 &= \sum_{k=0}^{n}(f_k^3 + f_k(f_k^2 - (-1)^k)) \\ &= \sum_{k=0}^{n}(2f_k^3 - f_k(-1)^k) \end{aligned}$$

and hence,

$$\begin{aligned} 2\sum_{k=0}^{n} f_k^3 &= f_nf_{n+1}^2 + \sum_{k=0}^{n} f_k(-1)^k \\ &= f_nf_{n+1}^2 + 1 + (-1)^n f_{n-1}, \end{aligned}$$

as desired.

□

Of course, a combinatorial purist prefers to avoid algebra altogether and the above argument can be modified to accommodate that desire. Similar to what we did in the proof of Identity 1, we establish an almost one-to-two correspondence between S (the set of triples of k-tilings as k ranges from 0 to n) and the set $V = \mathcal{F}_n \times \mathcal{F}_{n+1} \times \mathcal{F}_{n+1}$ consisting of all triples (A, B, C) of tilings with respective lengths n, $n+1$ and $n+1$.

For *almost* every triple (X, Y, Z) in $\mathcal{F}_k^3$, we wish to identify two triples in V. The first triple we generate depends on the parity of $n-k$. We map (X, Y, Z) to the triple

$$(A, B, C) = (Xd^{(n-k)/2}, Ysd^{(n-k)/2}, Zs^{n+1-k}),$$

when $n-k$ is even. (For example, the n-tiling A is equal to the k-tiling X followed by $(n-k)/2$ dominoes.) When $n-k$ is odd, we map (X, Y, Z) to the triple

$$(A, B, C) = (Xsd^{(n-1-k)/2}, Yd^{(n+1-k)/2}, Zs^{n+1-k}).$$

Notice that in both cases, (A, B, C) has parameter k and satisfies condition (i) of the previous proof. This mapping is well-defined for all elements of S and is easily reversed.

Next, for almost every triple (X, Y, Z) in $\mathcal{F}_k^3$, we generate a triple (A, B, C) with parameter k that satisfies condition (ii). Here we create, through tail-swapping, the tilings Y^+ and Z^- with respective lengths $k+1$ and $k-1$. These tilings exist unless Y and Z both consist entirely of dominoes. We now map this tiling to the triple

$$(A, B, C) = (Xd^{(n-k)/2}, Y^+d^{(n-k)/2}, Z^-ds^{n-k}),$$

when $n-k$ is even. When $n-k$ is odd, we let

$$(A, B, C) = (Y^+d^{(n-1-k)/2}, Xd^{(n+1-k)/2}, Z^-ds^{n-k}).$$

Notice that in both cases, (A, B, C) has parameter k with condition (ii), since the last domino of C covers cells k and $k+1$ and that (A, B, C) is unbreakable at every cell $j \geq k$. Given a tiling triple (A, B, C) with parameter k satisfying condition (ii), this mapping is easily reversed, except when Y^+ and Z^- consist of all dominoes.

To summarize what we have so far, this mapping is well-defined for all (X, Y, Z) in $\mathcal{F}_k$ except for those with even values of k where Y and Z consist of all dominoes. Almost balancing this out, however, this mapping hits all tiling triples (A, B, C) that satisfy condition (ii) except those with odd values of k for which Y^+ and Z^- consist of all dominoes. To pair most of these triples up, we extend our mapping so that for even values of k, we map the triple $(X, d^{k/2}, d^{k/2})$ to the triple

$$(A, B, C) = (d^{n/2}, \overline{X}d^{(n+1-\overline{k})/2}, d^{(\overline{k}+1)/2}s^{n-\overline{k}}),$$

when n is even, and to the triple

$$(A,B,C) = (\overline{X}d^{(n-\overline{k})/2}, d^{(n+1)/2}, d^{(\overline{k}+1)/2}s^{n-\overline{k}}),$$

when n is odd. As before, $\overline{X}$, is the conjugate of X with odd length $\overline{k}$.

This mapping is not defined when $k=0$, since Y^+ and Z^- do not exist. The only other situation where the mapping is undefined are those $(X,Y,Z) = (X, d^{n/2}, d^{n/2})$ in $\mathcal{F}_n^3$ where n is even and X ends with a square, since this would cause $\overline{k} = n+1$ and the triple (A,B,C) would therefore be undefined in its third component. Thus there are f_{n-1} triples (X,Y,Z) where this extended mapping is undefined. However, when n is even, all elements of V are hit, since $\overline{X}$ has length $\overline{k} \le n-1$, and so X can be recovered. Thus, when n is even, $2|S| - 1 - f_{n-1} = |V|$. That is, for even $n \ge 0$,

$$2\sum_{k=0}^{n} f_k^3 = f_n f_{n+1}^2 + f_{n-1} + 1,$$

as desired.

On the other hand, when n is odd, the mapping is always well-defined. Thus every non-empty triple (X,Y,Z) in S generates two triples in V. However, some elements of V are not mapped onto. Specifically, those $(A,B,C) = (\overline{X}, d^{(n+1)/2}, d^{(n+1)/2})$, where $\overline{X}$ has length $\overline{k} = n$ and ends with a square. Since there are f_{n-1} ways to create such an $\overline{X}$, we have $2|S| - 1 = |V| - f_{n-1}$, when n is odd. In other words, for odd $n \ge 1$,

$$2\sum_{k=0}^{n} f_k^3 = f_n f_{n+1}^2 - f_{n-1} + 1,$$

as desired.

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References

[1] A.T. Benjamin and J.J. Quinn, *Proofs That Really Count, The Art of Combinatorial Proof,* MAA, 2003.

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