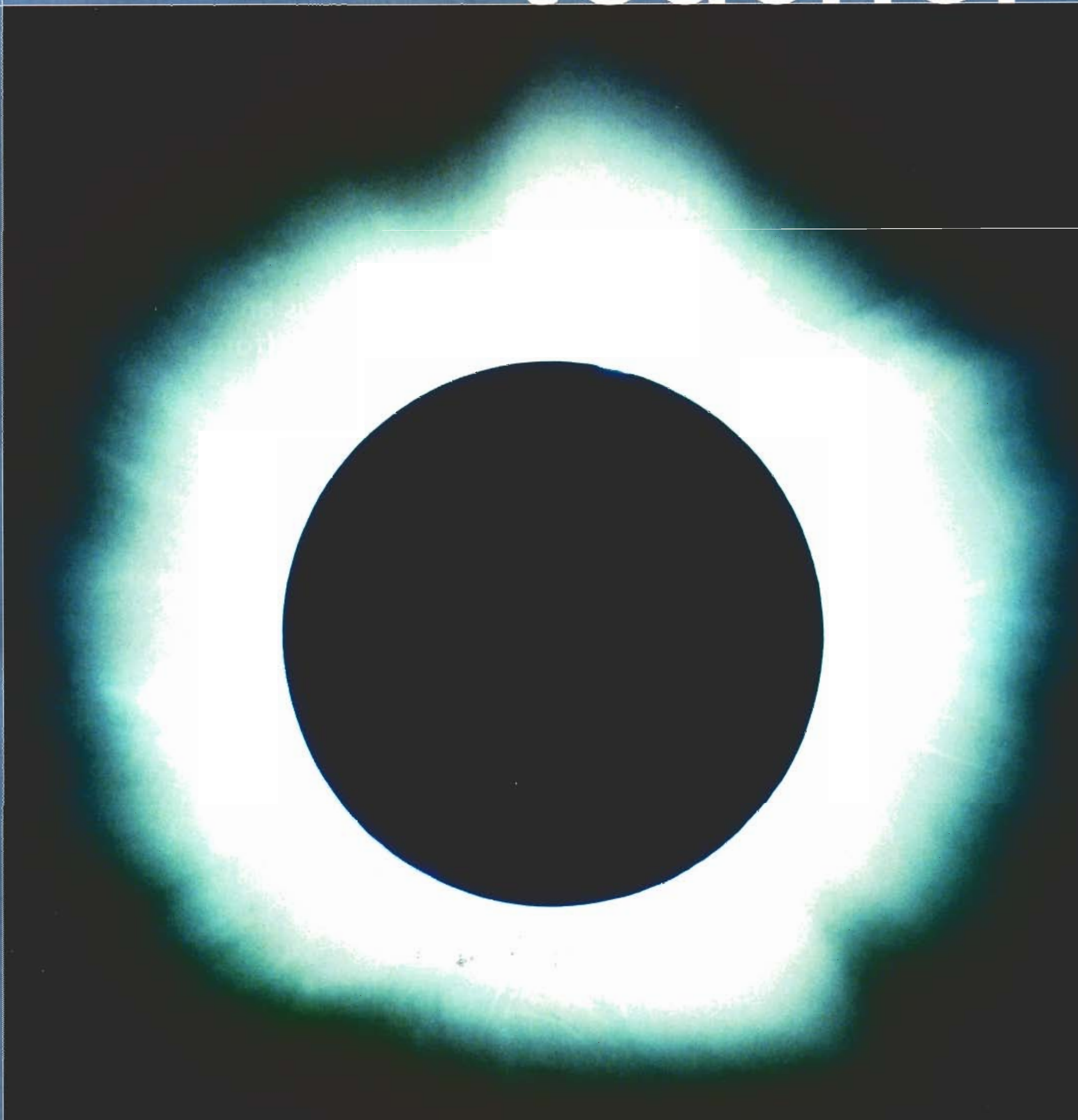
322 Alternate Graph Paper **328, 357** Fibonacci | VOLUME 99, NUMBER 5 | DECEMBER 2005/JANUARY 2006

MATHEMATICS teacher



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Arthur T. Benjamin and Jennifer J. Quinn

Revisiting Fibonacci and Related Sequences

We fully concur with Richard Askey's February 2004 "Delving Deeper" column. Discovering and proving identities containing Fibonacci numbers can be satisfying for students and teachers alike. His article touched on multiple strategies including induction, linear algebra, and a hefty dose of algebraic manipulation to derive many interesting identities. However, a single method can be employed to explain all of these identities more concretely, leading to deeper understanding and intuition. We are referring to the method of combinatorial proof.

A *combinatorial proof* explains an identity by counting—by counting a set in two different ways or by counting two sets and providing a correspondence between them. To apply this method, we need to know what the Fibonacci numbers *count*. We define the Fibonacci numbers by $f_0 = 1, f_1 = 1$, and for $n \geq 2, f_n = f_{n-1} + f_{n-2}$. Then the nth Fibonacci number counts the ways to tile a $1 \times n$ board with squares and dominoes. Letting s represent a square, which has length 1, and d represent a domino, which has length 2, $f_3 = 3$ counts the tilings

sss, sd, ds,

and $f_4 = 5$ counts the tilings

ssss, ssd, sds, dss, dd.

A tiling of length 5 (called a 5-tiling) can be created by appending a square to a 4-tiling or a domino to a 3-tiling. So there are

$$f_5 = f_4 + f_3 = 8$$

5-tilings. In general an n-tiling can be created by appending a square to an $(n - 1)$-tiling or a domino to an $(n - 2)$-tiling. Hence the number of tilings will continue to grow like the Fibonacci numbers.

Now consider the Fibonacci numbers and the squares of Fibonacci numbers given in **table 1**. Interestingly, the sum of the squares of two

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TABLE 1

n	0	1	2	3	4	5	6	7	8	9	10
f_n	1	1	2	3	5	8	13	21	34	55	89
f_n^2	1	1	4	9	25	64	169	441	1156	3025	7921

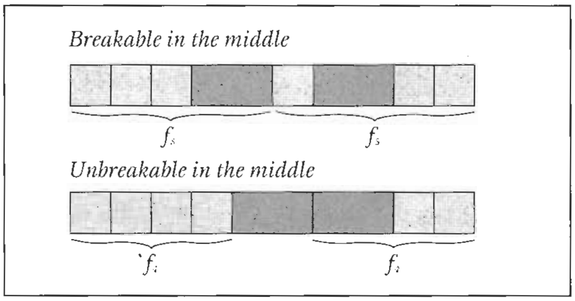

Fig. 1 A 10-tiling that is breakable in the middle can be cut into two tilings of length 5. A 10-tiling that is unbreakable in the middle must have a domino on cells 5 and 6; it can be cut into a 4-tiling, a domino, and another 4-tiling.

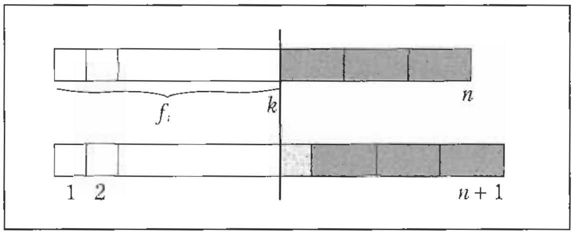

Fig. 2 There are f_k^2 tilings of an n-board and an $(n + 1)$-board with the last common break occuring at cell k.

consecutive Fibonacci numbers appears to be another Fibonacci number. More experimentation suggests that for all $n \geq 1$,

$$(1) \qquad f_{n-1}^2 + f_n^2 = f_{2n}.$$

This can be seen directly by a combinatorial argument that counts the set of tilings of length $2n$ in two different ways. On one hand, there are f_{2n} such tilings. On the other hand, either a $2n$-tiling is breakable in the middle or it is not, as illustrated in **figure 1**. A breakable tiling can be created in f_n^2 ways. An unbreakable tiling can be created in f_{n-1}^2 ways. Hence the set of $2n$-tilings is also counted by the sum

$$f_n^2 + f_{n-1}^2.$$

However, there is nothing special about the middle of an even length board. We can take this idea further by investigating the breakability after *any* cell. This time let's count the tilings of length $m + n$ in two different ways. There are certainly f_{m+n} such tilings. But now ask whether the tiling is breakable after cell m. Here, a breakable tiling can be created in $f_m f_n$ ways. An unbreakable tiling has a domino covering cells m and $m + 1$; it can be created in $f_{m-1}f_{n-1}$ ways. So

$$(2) \qquad f_{m+n} = f_m f_n + f_{m-1} f_{n-1},$$

generalizing the Fibonacci pattern given in equation (1).

Now consider the partial sum of the squares of the Fibonacci numbers. For example:

$$\begin{aligned} 1^2 + 1^2 &= 2 = 1 \cdot 2 \\ 1^2 + 1^2 + 2^2 &= 6 = 2 \cdot 3 \\ 1^2 + 1^2 + 2^2 + 3^2 &= 15 = 3 \cdot 5 \\ 1^2 + 1^2 + 2^2 + 3^2 + 5^2 &= 40 = 5 \cdot 8 \end{aligned}$$

The pattern suggests that

$$f_0^2 + f_1^2 + \cdots + f_n^2 = f_n \cdot f_{n+1}.$$

To "see" why this is true, we count the ways to tile a board of length n and another board of length $n + 1$. Naturally, this can be done in $f_n f_{n+1}$ ways. How many of these tilings have their last common break, called a *fault*, at cell k? From **figure 2** we see the answer is f_k^2 since after the last fault, there is just one way to tile the two boards in a fault-free way. The index k can be as small as 0, since both tilings have a common fault before the first cell, and k can be as large as n, when the second tiling ends with a square. Summing over all values of k shows that there are

$$\sum_{k=0}^{n} f_k^2 \text{ tilings,}$$

as desired.

As an exercise, we invite the reader to show that

$$f_0 + f_1 + \cdots + f_n = f_{n+2} - 1$$

by counting the tilings of a board of length $n + 2$ with at least one domino, and considering the location of the last domino. Additionally, by tiling boards of length $2n$ or $2n + 1$ and considering the location of the last square, derive

$$f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1}$$

and

$$f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n} - 1.$$

Returning to **table 1**, we see that the square of a Fibonacci number and the product of its neighboring Fibonacci numbers always differs by 1. For example, $5^2 - 3 \cdot 8 = 1$, $8^2 - 5 \cdot 13 = -1$, and in general it appears that

$$(3) \qquad f_n^2 - f_{n-1} f_{n+1} = (-1)^n.$$

To understand this identity, we illustrate that every pair of n-tilings can be easily transformed into two tilings of length $n + 1$ and length $n - 1$.

Suppose A and B are n-tilings where A covers cells 1 through n and B covers cells 2 through $n + 1$, as shown in **figure 3**. Suppose A and B have their last fault at cell k. Then A can be decomposed into a k-tiling A_1, followed by an $n - k$ tiling A_2. Similarly B can be decomposed into a $(k - 1)$-tiling B_1, followed by an $(n - k + 1)$-tiling B_2. We exchange A_2 and B_2 (a process called *tailswapping*) to obtain the tilings $A^+ = A_1B_2$ and $B^- = B_1A_2$, shown in **figure 4**, with length $n + 1$ and $n - 1$, respectively. Notice that tailswapping A^+ and B^- in **figure 4** brings us back to A and B. Hence there are *almost* as many tiling pairs (A, B) where A and B have length n as there are tiling pairs (C, D) where C has length $n + 1$ and D has length $n - 1$. We say *almost* because the tiling that consists of all dominoes is the only *one* that is fault-free. Since (A, B) can be fault-free only when n is even and (C, D) can be fault-free only when n is odd, we understand why

$$f_n^2 - f_{n-1}f_{n+1} = (-1)^n.$$

Delving deeper into the technique of tailswapping easily allows us to discover and prove more general identities. For example, if A and B are n-tilings with an offset of r (instead of 1) cells, then tailswapping produces A^+ and B^- with respective lengths $n + r$ and $n - r$. Thus f_n^2 is *almost* $f_{n-r}f_{n+r}$. Their difference is the number of fault-free tilings. The pair (A^+, B^-) can only be fault-free when $n - r$ is even, B^- has all dominoes, and A^+ has all dominoes on cells r through $n + 1$. The remaining cells of A^+ (1 through $r - 1$ and $n + 2$ through $n + r$) can be tiled f_{r-1}^2 ways. On the other hand (A, B) has f_{r-1}^2 fault-free tilings precisely when $n - r$ is odd. Can you find them? Consequently

$$f_n^2 - f_{n-r}f_{n+r} = (-1)^{n+1-r} f_{r-1}^2.$$

By changing the picture slightly and allowing nonsymmetrical offsets as in **figure 5** we can easily "see" the following identity from Askey (2004):

$$f_{n+s}f_{n-r} - f_n f_{n+s-r} = (-1)^{n-r} f_{r-1}f_{s-1}.$$

Of course it is no surprise that combinatorial proofs can be given to identities that involve binomial coefficients. For example, the "diagonal sum of Pascal's Triangle" identity,

$$\sum_{k=0}^{n/2} \binom{n-k}{k} = f_n,$$

counts length n tilings by considering how many dominoes are used. A length n tiling with k dominoes must have $n - 2k$ squares and therefore has $k + (n - 2k) = n - k$ tiles, which from left to right are named Tile 1, Tile 2, . . . , Tile $n - k$. Since there are

$$\binom{n-k}{k}$$

ways to choose k of these tiles to be dominoes, the identity follows.

But it may come as a surprise to see other Fibonacci identities proved this way. Recall the Euclidean algorithm for finding the greatest common divisor of two numbers. It is based on the fact that if $n = qm + r$, then

$$\gcd(n, m) = \gcd(m, r),$$

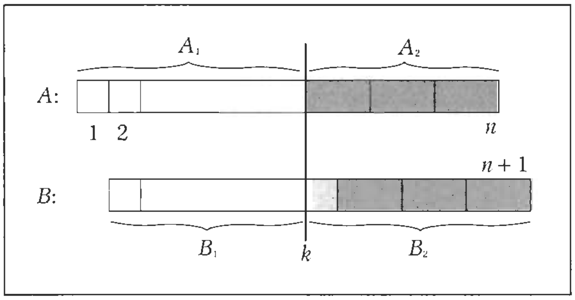

Fig. 3 Two n-tilings A and B, offset by one cell, with their last fault occurring at cell k

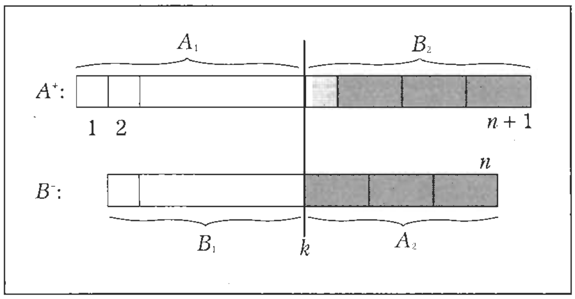

Fig. 4 Swapping the tails of the two n-tilings in figure 3, i.e., the subtilings A_2 and B_2, creates an $(n + 1)$-tiling A^+ and an $(n - 1)$-tiling B^-. This provides a bijection between offset tiling pairs with faults.

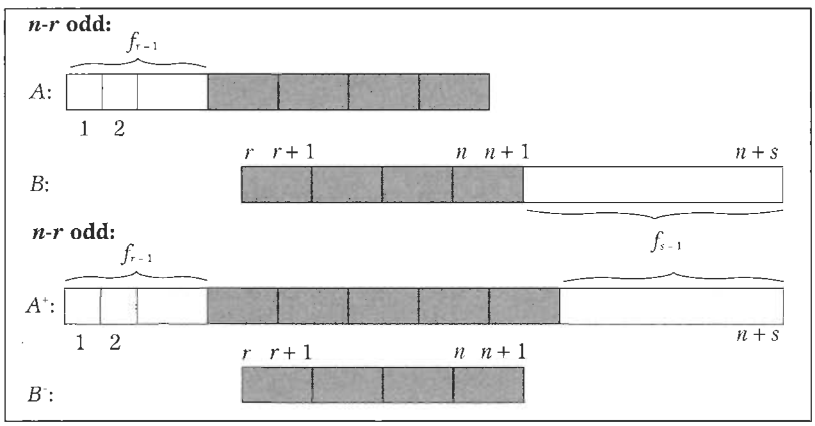

Fig. 5 An n-tiling and an $(n + s - r)$-tiling with asymmetric offsets can use tail swapping to establish the identity $f_{n+s}f_{n-r} - f_nf_{n+s-r} = (-1)^{n-r}f_{r-1}f_{s-1}$. The tilings that cannot be swapped are pictured here.

since $\gcd(n, m)$ is unchanged by subtracting a multiple of m from n. For example, if we always choose q to be as large as possible, then we can compute

$$\gcd(978, 96) = \gcd(96, 18) = \gcd(18, 6) = \gcd(6, 0) = 6.$$

One of our favorite Fibonacci fun facts is best described by the traditional definition of Fibonacci numbers whereby $F_0 = 0$, $F_1 = 1$ and in general, for $n \geq 0$, $F_n = f_{n-1}$. Behold,

$$\gcd(F_n, F_m) = F_{\gcd(n, m)}. \tag{4}$$

We have already proved special cases of this equation. For instance, equation (3) demonstrates that consecutive Fibonacci numbers are relatively prime since any number that divides F_n and F_{n+1} must also divide ± 1. (This fact can also be proved by induction.) Thus:

$$\gcd(F_{n+1}, F_n) = 1 = F_1 = F_{\gcd(n+1, n)}$$

Also observe that if n is a multiple of m, say $n = qm$, then by equation (2):

$$\begin{aligned} F_n = F_{qm} = f_{qm-1} &= f_{(m-1)+(q-1)m} \\ &= f_{m-1}f_{(q-1)m} + f_{m-2}f_{(q-1)m-1} \\ &= F_m F_{(q-1)m+1} + F_{m-1}F_{(q-1)m} \end{aligned}$$

Thus if $F_{(q-1)m}$ is a multiple of F_m, then so is F_{qm}. So by induction on q, we have that $F_n = F_{qm}$ is a multiple of F_m. (For an inductionless proof, see Benjamin and Quinn, June 2003, or Benjamin and Quinn 2003.) Therefore, if $n = qm$, then

$$\gcd(F_n, F_m) = \gcd(F_{qm}, F_m) = F_m = F_{\gcd(n, m)}.$$

Now suppose that $n = qm + r$ where $r > 0$. Then by equation (3),

$$\begin{aligned} \gcd(F_n, F_m) &= \gcd(F_m, F_{qm+r}) = \gcd(F_m, f_{qm-1+r}) \\ &= \gcd(F_m, f_{qm-1}f_r + f_{qm-2}f_{r-1}) \\ &= \gcd(F_m, F_{qm}F_{r+1} + F_{qm-1}F_r) \\ &= \gcd(F_m, F_{qm-1}F_r) \end{aligned}$$

where the last step follows, since F_{qm} is a multiple of F_m and thus $F_{qm}F_{r+1}$ can be ignored when computing the greatest common divisor with F_m. Also, we know that F_m has no factors in common with F_{qm-1} (since F_m divides F_{qm}, which is relatively prime to F_{qm-1}) and therefore

$$\gcd(F_n, F_m) = \gcd(F_m, F_r)$$

But wait. This is just the Euclidean algorithm with Fs inserted on top of everything. Let's call this the FEuclidean FAlgorithm.

Consequently, to find $\gcd(F_{978}, F_{96})$, FEuclid's FAlgorithm finds

$$\begin{aligned} \gcd(F_{978}, F_{96}) &= \gcd(F_{96}, F_{18}) = \gcd(F_{18}, F_6) \\ &= \gcd(F_6, F_0) = F_6, \end{aligned}$$

since $F_0 = 0$. In general, if $\gcd(n, m) = g$, then Euclid's algorithm starts with $\gcd(n, m)$ and eventually reduces it to $\gcd(g, 0) = g$. Thus FEuclid's FAlgorithm begins with $\gcd(F_n, F_m)$ and eventually reduces it to $\gcd(F_g, F_0) = F_g$. Thus for any n and m, $\gcd(F_n, F_m) = F_{\gcd(n, m)}$ as promised.

We have only barely scratched the surface of what is possible with combinatorial proofs. Going further, if we allow c_1 colors for squares and c_2 colors for dominoes, we can combinatorially explain identities defined by initial conditions $u_j = 0$ for $j < 0$, $u_0 = 1$ and by the recurrence $u_n = c_1u_{n-1} + c_2u_{n-2}$, for $n \geq 1$. Identities for kth order recurrences $u_n = c_1u_{n-1} + \cdots + c_ku_{n-k}$ with the same initial conditions are also easily explained by allowing colored tiles with length at most k. Recurrences with other initial conditions can be modeled by the same tiling problems, but restrictions are placed on the initial tile. For details, see Benjamin and Quinn (June 2003) or Benjamin and Quinn (2003). Finally, by looking at random tilings, even identities that involve the golden ratio

$$\phi = \frac{1+\sqrt{5}}{2}$$

such as Binet's formula

$$F_n = \frac{1}{\sqrt{5}}\left[\phi^n - (-1/\phi)^n\right]$$

can be given a combinatorial explanation.

We end this article with a comment from Richard Askey, who was quoted in the preface of Boros and Moll (2004) as saying,

If things are nice, there is probably a good reason why they are nice: and if you do not know at least one reason for this good fortune, then you still have work to do.

So the next time you see a nice identity like

$$\sum_{i\geq 0}\sum_{j\geq 0}\binom{n-i}{j}\binom{n-j}{i} = f_{2n+1}$$

or

$$\sum_{k=0}^{n}(n-k)f_k = f_{n+3} - (n+3),$$

we hope you will ask yourself, "What is the underlying combinatorial explanation?"

At least that's what we are counting on.

Editors' notes: Arthur Benjamin and Jennifer Quinn add one more approach to the growing storehouse of methods we have seen in "Delving Deeper" for solving recurrence relations. This approach involves combinatorial proof, what they call *Proofs That Really Count* (Benjamin and Quinn 2003). We plan an installment sometime next year to summarize and catalogue the methods that have appeared so far. Send us your favorites.

Benjamin and Quinn include some provocative questions. We have a few more to add. The context of tiling a wall leads to many interesting counting problems.

- The authors describe how, if "triominoes" (tiles of length 3) are allowed in the tiling, the number of ways there are to tile a length n wall satisfies the recurrence

  $$u_n = u_{n-1} + u_{n-2} + u_{n-3}$$

  What are the initial conditions? What properties do these numbers have?

- What if tiles of any length are allowed in the tiling? How many ways are there to tile a length n wall? Such tilings are known as compositions of n (see Benjamin and Quinn 2003 for more details). Many teachers know this as the Trains problem (Parker 1991): Count the number of trains of length n that can be made from Cuisinaire-like rods. There are many related questions. For example:

  (1) How many Cusineaire trains of length n have exactly k cars?
  (2) Suppose you lay out all the Cusineaire trains of length n on your desk. How many rods of length k will there be on the desk? (Connie Vann, a teacher in Danvers, Massachusetts, came up with this problem.)

There are many other questions one could ask. Why not pick one and work on it?

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