# Strong Chromatic Index of Subset Graphs

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## ABSTRACT

The strong chromatic index of a graph $G$, denoted $sq(G)$, is the minimum number of parts needed to partition the edges of $G$ into induced matchings. For $0 \le k \le l \le m$, the subset graph $S_m(k,l)$ is a bipartite graph whose vertices are the $k$- and $l$-subsets of an $m$ element ground set where two vertices are adjacent if and only if one subset is contained in the other. We show that $sq(S_m(k,l)) = \binom{m}{l-k}$ and that this number satisfies the strong chromatic index conjecture by Brualdi and Quinn for bipartite graphs. Further, we demonstrate that the conjecture is also valid for a more general family of bipartite graphs. 

## INTRODUCTION

In a proper edge coloring of a graph, $G$, no two incident edges are given the same color. Thus any path in $G$ connecting two edges of the same color must have length at least 1. A coloring is called a *strong edge coloring* if any path in $G$ connecting two edges of the same color has length at least 2. In other words, the subgraph of $G$ induced by edges of the same color forms a matching in $G$. The *strong chromatic index*, $sq(G)$, equals the smallest number of colors in any strong edge coloring.

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Erdős and Nešetřil [4] conjecture that $sq(G)$ is at most $\frac{5}{4}\Delta^2$, where $\Delta$ denotes the maximum degree of a vertex in $G$. For a bipartite graph $G = (X, E, Y)$, Brualdi and Quinn [2] conjecture

$$sq(G) \leq \Delta(X)\Delta(Y), \tag{0.1}$$

where $\Delta(X)$ and $\Delta(Y)$ are the maximum degrees in their respective parts.

As with most coloring problems, there are only a few classes of graphs whose strong chromatic index is known exactly. Complete graphs and complete bipartite graphs require all edges to be colored differently, i.e., $sq(K_n) = \binom{n}{2}$ and $sq(K_{m,n}) = mn$. In any antimatching of $G$ (that is to say, a subgraph with no induced matching of size greater than 1) all edges must be given a different color. Therefore the strong chromatic index of $G$ is greater than or equal to the number of edges in the largest antimatching in $G$. This result holds with equality for chordal graphs [3] and trees [5]. Specifically, for a tree, $T, sq(T) = \max_{x \sim y}\{\deg(x) + \deg(y) - 1\}$. Faudree et al. [5] also compute exact strong chromatic indices for $n$-dimensional cubes, Kneser graphs, and revolving door graphs. In this paper, we will define a family of graphs called subset graphs for which the strong chromatic index can be determined exactly.

## SUBSET GRAPHS

For integers $0 \leq k \leq l \leq m$, we define the *subset graph*, $S_m(k, l)$ to be the bipartite graph $(\mathcal{X}, E, \mathcal{Y})$ where the vertices of $\mathcal{X}$ are the $k$-subsets of $[m] = \{1, 2, \ldots, m\}$, the vertices of $\mathcal{Y}$ are the $l$-subsets of $[m]$, and for $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, $X$ is adjacent to $Y$ if and only if $X \subseteq Y$. Notice that the subset graph $S_m(k, k)$ is a matching with $\binom{m}{k}$ edges. Also the revolving door graph of order $d$ is the subset graph $S_{2d-1}(d-1, d)$. See Figure 1.

**Theorem 1.** $sq(S_m(k, l)) = \binom{m}{l-k}$.

***Proof.*** First we construct a strong edge coloring of $S_m(k, l)$ using exactly $\binom{m}{l-k}$ colors. The *colors* are precisely the $(l-k)$-subsets of $[m]$. For each edge $\{X, Y\}$ we assign the color $Y \setminus X$. To show that the coloring is strong, suppose that the edges $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$ are assigned the same color, say $C$, and that $X_1$ is adjacent to $Y_2$. Since $C$ is a subset of $Y_2$ and is disjoint from $X_1$, then $C$ is a subset of $Y_2 \setminus X_1$. But $|C| = |Y_2 \setminus X_1|$, so the color assigned to the edge $\{X_1, Y_2\}$ is $C$. Consequently $Y_1 = X_1 + C = Y_2$ and $X_1 = Y_2 \setminus C = X_2$. Hence if $\{X_1, Y_1\}$

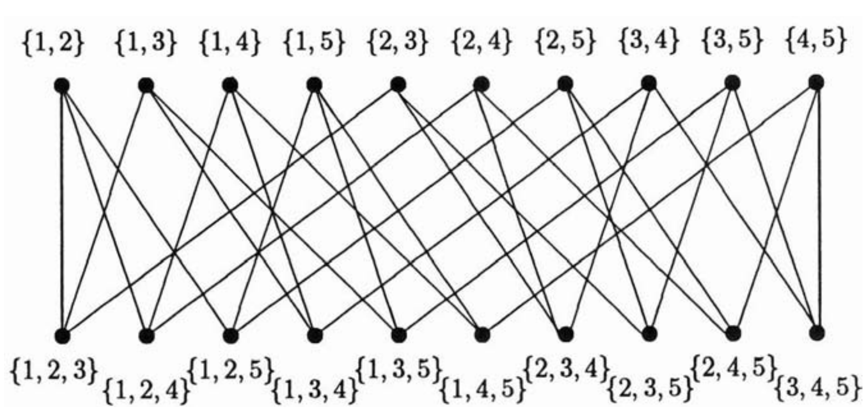


FIGURE 1. The subset graph $S_5(2, 3)$.

and $\{X_2, Y_2\}$ are distinct edges assigned the same color, then $X_1$ and $Y_2$ are not adjacent. Thus the coloring is strong and $sq(S_m(k,l)) \le \binom{m}{l-k}$.

We show that equality holds by constructing an antimatching, $A \subseteq E$, of size $\binom{m}{l-k}$. We let the edge $\{X, Y\}$ be an element of $A$ if and only if there exists an integer $j$ for which $X \subseteq [j] \subseteq Y$. To see that $A$ contains no induced matching, suppose $X_1 \subseteq [j_1] \subseteq Y_1$ and $X_2 \subseteq [j_2] \subseteq Y_2$ where $j_1 \le j_2$. Then $X_1 \subseteq [j_1] \subseteq [j_2] \subseteq Y_2$ implies $\{X_1, Y_2\}$ is also in $A$; thus $A$ is an antimatching.

Now we count the edges in $A$. For each edge $\{X, Y\}$ in $A$ there is a unique $j$ for which $X \subseteq [j] \subseteq Y$ and $j+1 \notin Y$. For a given $j$, the number of edges $\{X, Y\}$ satisfying this is $\binom{j}{k}\binom{m-(j+1)}{l-j}$. Summing over the nonzero terms gives

$$|A| = \sum_{j=k}^{l} \binom{j}{k}\binom{m-(j+1)}{l-j} = \binom{m}{l-k}.$$

The identity expressed in the last equality can be viewed as counting the number of length $m$ paths from (0, 0) to $(m-(l-k), l-k)$ by conditioning on those paths which go through the points $(k, j-k)$ and $(k+1, j-k)$ with $j = k, k+1, \ldots, l$. Hence $sq(S_m(k,l)) \ge |A| = \binom{m}{l-k}$. ∎

A family of graphs related to $S_m(k,l) = (\mathcal{X}, E, \mathcal{Y})$ can be defined using the same vertex set. Let the *disjoint subset graph* $D_m(k,l) = (\mathcal{X}, E', \mathcal{Y})$, where $\{X, Y\}$ is an edge of $E'$ if and only if $X \cap Y = \emptyset$. A disjoint subset graph can be considered a bipartite version of a Kneser graph.

**Corollary.** $sq(D_m(k,l)) = \binom{m}{l+k}$.

***Proof.*** Since $X \cap Y = \emptyset$ if and only if $X$ is a subset of $[m] \setminus Y$, then $D_m(k,l)$ is isomorphic to $S_m(k, m-l)$. Theorem 1 implies

$$sq(D_m(k,l)) = \binom{m}{(m-l)-k} = \binom{m}{l+k}.$$

Note that if $k > m-l$, an empty intersection is impossible and $D_m(k,l)$ has no edges. Thus $sq(D_m(k,l)) = 0 = \binom{m}{l+k}$. ∎

## UPPER BOUNDS

In this section, we show that subset graphs satisfy the conjectured inequality for the strong chromatic index of bipartite graphs given in (0.1). Further, we demonstrate that the inequality is valid for a more general family of bipartite graphs.

**Theorem 2.** The subset graph $S_m(k,l) = (\mathcal{X}, E, \mathcal{Y})$ satisfies

$$sq(S_m(k,l)) \le \Delta(\mathcal{X})\Delta(\mathcal{Y}).$$

***Proof.*** Every vertex $X \in \mathcal{X}$ has degree $\binom{m-k}{l-k}$ and every vertex $Y \in \mathcal{Y}$ has degree $\binom{l}{k}$. By Theorem 1, we must show that

$$\binom{m}{l-k} \le \binom{m-k}{l-k}\binom{l}{k}. \tag{0.2}$$

Substituting $j = l - k$, our inequality becomes

$$\binom{m}{j} \le \binom{m-k}{j}\binom{k+j}{k}. \tag{0.3}$$

Let $\mathcal{P}$ be the set of length $m$ paths from (0, 0) to $(m-j, j)$ and let $\mathcal{P}'$ be the set of length $m+j$ paths from (0, 0) to $(m-j, 2j)$ which pass through the point $(k, j)$. Since $|\mathcal{P}| = \binom{m}{j}$ and $|\mathcal{P}'| = \binom{k+j}{k}\binom{m-k}{j}$, we prove (0.3) by constructing a one-to-one mapping from $\mathcal{P}$ to $\mathcal{P}'$. For any path $P \in \mathcal{P}$, let $P'$ be the path obtained by inserting $j$ vertical steps prior to the $(k+1)$st horizontal step in $P$ (see Figure 2). The path $P'$ contains $m+j$ steps, passes through the point $(k, j)$, and terminates at the point $(m-j, 2j)$. Hence $P' \in \mathcal{P}'$. Notice that $P'$ coincides with $P$ through the $k$th horizontal step and parallels $P$ beginning with the $k+1$st horizontal step. This mapping is one-to-one and hence inequality (0.3) follows. ∎

Now we consider the following generalization of subset graphs. Define $S_m(k, l, \lambda) = (\mathcal{X}, E, \mathcal{Y})$ where the vertices of $\mathcal{X}$ are the $k$-subsets of $[m]$, the vertices of $\mathcal{Y}$ are the $l$-subsets of $[m]$, and for $X \in \mathcal{X}$, and $Y \in \mathcal{Y}, X$ is adjacent to $Y$ if and only if $|X \cap Y| = \lambda$. Note that subset graphs and disjoint subset graphs are special cases, corresponding to $\lambda = k$ and $\lambda = 0$, respectively. Exact values for the strong chromatic index of these graphs are presently unknown, however we present two strong edge colorings which together satisfy inequality (0.1).

**Lemma 3.** $sq(S_m(k, l, \lambda)) \le \binom{m}{k+l-2\lambda}\binom{m-(k+l-2\lambda)}{\lambda}$.

***Proof.*** We partition the edge set $E$ of $S_m(k, l, \lambda)$ according to the symmetric difference and intersection of the vertices. Specifically, given disjoint sets $D \subseteq [m]$ and $I \subseteq [m]$, where $|D| = k + l - 2\lambda$ and $|I| = \lambda$, let

$$E_{D,I} = \{\{X, Y\} \in E \mid X \oplus Y = D \text{ and } X \cap Y = I\},$$

where $X \oplus Y = X \setminus Y \cup Y \setminus X$. We claim that $E_{D,I}$ is an induced matching in $S_m(k, l, \lambda)$, and therefore its edges can be assigned the same color in a strong edge coloring. To see this, suppose

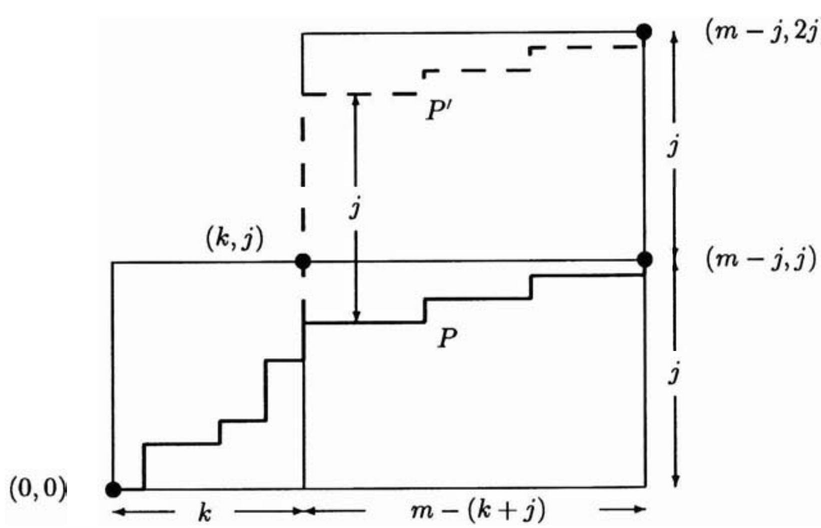


FIGURE 2. $\binom{m}{j} \le \binom{m-k}{j}\binom{k+j}{k}$

that $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$ are edges of $E_{D,I}$ and that $\{X_1, Y_2\} \in E$. Then $|X_1 \cap Y_2| = \lambda$, so their intersection must be $I$. Also, since $Y_2 \subseteq X_2 \cup Y_2 = D \cup I = X_1 \cup Y_1$, it follows that $X_1 \cup Y_2$ is a subset of $X_1 \cup Y_1$, and since both have the same size, they must be equal. Further $X_1 \cap Y_2 = I$ implies that $X_1 \oplus Y_2 = D$. Hence $Y_2 = (D \setminus X_1) \cup I = Y_1$ and $X_1 = (D \setminus Y_2) \cup I = X_2$, and $E_{D,I}$ is an induced matching.

For any edge $\{X, Y\} \in E$, the symmetric difference of $X$ and $Y$ has size $k + l - 2\lambda$. Thus $D$ and $I$ can be chosen in $\binom{m}{k+l-2\lambda}\binom{m-(k+l-2\lambda)}{\lambda}$ ways, providing us with the desired upper bound. ∎

The second strong edge coloring is similar to the decomposition above, but it partitions the edges according to the "differences" of the vertices.

**Lemma 4.** $sq(S_m(k,l,\lambda)) \le \binom{m}{k+l-2\lambda}\binom{k+l-2\lambda}{k-\lambda}$.

***Proof.*** Given disjoint sets $D_X \subseteq [m]$ and $D_Y \subseteq [m]$, where $|D_X| = k - \lambda$ and $|D_Y| = l - \lambda$, let

$$E_{D_X,D_Y} = \{\{X, Y\} \in E \mid X \setminus Y = D_X \text{ and } Y \setminus X = D_Y\}$$

We claim that $E_{D_X,D_Y}$ is an induced matching in $S_m(k,l,\lambda)$. Suppose that $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$ are edges of $E_{D_X,D_Y}$ and that $\{X_1, Y_2\} \in E$. Then $|X_1 \cap Y_2| = \lambda$. Thus $|X_1 \setminus Y_2| = k - \lambda = |D_X|$. But since $X_1 \setminus Y_1 = D_X = X_2 \setminus Y_2$, we must have $D_X \subseteq X_1 \setminus Y_2$, and therefore $D_X = X_1 \setminus Y_2$. Analogously, $D_Y = Y_2 \setminus X_1$. But this implies

$$\begin{aligned} X_1 &= (X_1 \setminus Y_2) \cup (X_1 \cap Y_2) = D_X \cup [Y_2 \setminus (Y_2 \setminus X_1)] \\ &= D_X \cup (Y_2 \setminus D_Y) = (X_2 \setminus Y_2) \cup (Y_2 \cap X_2) = X_2. \end{aligned}$$

Similarly, $Y_2 = Y_1$, and $E_{D_X,D_Y}$ is an induced matching. There are $k + l - 2\lambda$ elements in $D_X \cup D_Y$, so $D_X$ and $D_Y$ can be chosen in $\binom{m}{k+l-2\lambda}\binom{k+l-2\lambda}{k-\lambda}$ ways. ∎

We combine Lemmas 3 and 4 to obtain the following result.

**Theorem 5.** The graph $S_m(k,l,\lambda) = (\mathcal{X}, E, \mathcal{Y})$ satisfies

$$sq(S_m(k,l,\lambda)) \le \Delta(\mathcal{X})\Delta(\mathcal{Y}).$$

***Proof.*** The degree of any vertex in $\mathcal{X}$ is $\binom{k}{\lambda}\binom{m-k}{l-\lambda}$, and the degree of any vertex in $\mathcal{Y}$ is $\binom{l}{\lambda}\binom{m-l}{k-\lambda}$. Hence our theorem is equivalent to proving

$$sq(S_m(k,l,\lambda)) \le \binom{k}{\lambda}\binom{m-k}{l-\lambda}\binom{l}{\lambda}\binom{m-l}{k-\lambda}.$$

By Lemmas 3 and 4, it suffices to show that

$$\binom{m}{k+l-2\lambda}\min\left\{\binom{m-(k+l-2\lambda)}{\lambda}, \binom{k+l-2\lambda}{k-\lambda}\right\} \le \binom{k}{\lambda}\binom{m-k}{l-\lambda}\binom{l}{\lambda}\binom{m-l}{k-\lambda}. \quad (0.4)$$

Fortunately in [1], it is shown that for any nonnegative integers $a, b, c$, and $d$

$$\binom{a+b+c+d}{a+b}\min\left\{\binom{a+b}{a},\binom{c+d}{c}\right\} \leq \binom{a+c}{a}\binom{a+d}{a}\binom{b+c}{b}\binom{b+d}{b}. \quad (0.5)$$

The desired inequality follows by setting $a = k - \lambda, b = l - \lambda, c = \lambda$, and $d = m - k - l + \lambda$. ∎

We note that equality holds in (0.5) if and only if at least two of $a, b, c$, or $d$ are zero. Hence the only graphs for which (0.4) holds with equality are matchings and graphs of the form $K_{1,t}$. The strong chromatic indices of these graphs are 1 and $t$, respectively.

Also, note that inequality (0.4) reduces to inequality (0.2) when $\lambda = k$ and to $\binom{m}{k+l} \leq \binom{m-k}{l}\binom{m-l}{k}$ when $\lambda = 0$. These are the inequalities needed to directly establish that the strong chromatic index is less than or equal to $\Delta(\mathcal{X})\Delta(\mathcal{Y})$ for subset graphs and disjoint subset graphs respectively.

Interestingly, neither Lemma 3 nor Lemma 4 alone is strong enough to prove Theorem 5 since for each construction method, there exist parameters $\lambda \leq k \leq l \leq m$ for which the number of colors used exceeds $\Delta(\mathcal{X})\Delta(\mathcal{Y})$. For instance, when $m = 2, l = k = 1$, and $\lambda = 1$ ($\lambda = 0$), the construction of Lemma 3 (Lemma 4) uses 2 colors while $\Delta(\mathcal{X})\Delta(\mathcal{Y}) = 1$.

## FURTHER WORK

There are many properties of subset graphs yet to be explored. We would like to determine $sq(S_m(k, l, \lambda))$ exactly for $0 < \lambda < k$. It may also be possible to determine a *stronger* chromatic index, $s^d q(G)$, for subset graphs. We define $s^d q(G)$ to be the minimum number of colors in an edge coloring of $G$ such that any path in $G$ connecting two edges of the same color has length greater than $d$. Using this notation, note that $s^0 q(G)$ and $s^1 q(G)$ are the chromatic index and the strong chromatic index respectively.

Ultimately, we would like to prove that inequality (0.1) holds for all bipartite graphs. To establish this, we need only consider bipartite graphs which are *biregular* (i.e., all vertices in the same part have the same degree) since it can be shown that any bipartite graph can be embedded in a biregular bipartite graph with the same maximum degrees. (This follows from the Gale-Ryser Theorem on the existence of zero-one matrices with given line sums, see e.g. [7].) Since subset graphs are biregular, our hope is that they will be a useful tool to show that inequality (0.1) holds.

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