Sums of Evenly Spaced Binomial Coefficients

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Two of the first identities encountered in a discrete mathematics course are the following finite sums of binomial coefficients. For $n \geq 0$,

$$\sum_{k\geq 0} \binom{n}{k} = 2^n \tag{1}$$

and for $n \geq 1$,

$$\sum_{k\geq 0} \binom{n}{2k} = 2^{n-1}. \tag{2}$$

The sums are finite since $\binom{n}{k} = 0$ when $k > n$. Both of these identities have elementary combinatorial proofs. But when $r \geq 3$, the sum $\sum_{k\geq 0} \binom{n}{rk}$ is rarely mentioned because its closed form is more *complex*. (See Gould [**1**]. A special case appears in [**3**] as problem 1.42(f).)

THEOREM 1. *For $n \geq 0$ and $r \geq 1$,*

$$\sum_{k\geq 0} \binom{n}{rk} = \frac{1}{r}\sum_{j=0}^{r-1}(1+\omega^j)^n, \tag{3}$$

where $\omega = e^{i2\pi/r}$ is a primitive rth root of unity.

Notice that when $r = 1$ or 2, we have $\omega = 1$ or -1 respectively and the formulas in equations (1) and (2) are directly obtained. When $r = 3$, we have $\omega = e^{i2\pi/3} = \frac{-1+\sqrt{3}i}{2}$ and then Theorem 1 yields, for $n \geq 0$,

$$\sum_{k\geq 0} \binom{n}{3k} = \frac{2^n + m}{3},$$

where m depends on n and is equal to $2, 1, -1, -2, -1, 1$, when n is congruent to $0, 1, 2, 3, 4, 5 \pmod 6$ respectively. Likewise when $r = 4$, we have $\omega = i$, and we get

$$\sum_{k\geq 0}\binom{n}{4k} = \frac{2^n + m2^{\lceil n/2\rceil}}{4},$$

where $m = 2, 1, 0, -1, -2, -1, 0, 1$, when $n \geq 1$ is congruent, respectively, to $0, 1, 2, 3, 4, 5, 6, 7 \pmod 8$. (When $n = 0$, this formula needs to be adjusted, since $0^0 = 1$.)

A generalization of Theorem 1 (which appears in Gould [**1**] in modified form) also has an attractive closed form.

THEOREM 2. *For any integers $0 \leq a < r$ and $n \geq 0$,*

$$\sum_{k\geq 0}\binom{n}{a+rk} = \frac{1}{r}\sum_{j=0}^{r-1}\omega^{-ja}(1+\omega^j)^n, \tag{4}$$

where $\omega = e^{i2\pi/r}$ is a primitive rth root of unity.

While Theorems 1 and 2 have succinct algebraic explanations using the binomial theorem (see [**2**], [**3**]), our goal is to prove them combinatorially. In a combinatorial proof, an identity is proved by counting a problem in two different ways. Our proofs will utilize the graph C_r, the *directed, looped cycle graph* with vertex set $V = \{0, 1, \ldots, r-1\}$ such that for each vertex j, there is an arc to vertex j and $j+1 \pmod r$. (See Figure 1.) We define an n-walk to be a walk on C_r that takes

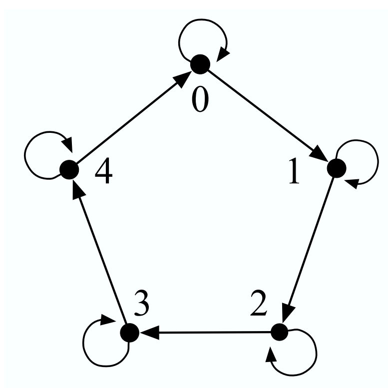

FIGURE 1: The looped cycle graph C_5.

exactly n steps. A walk that begins and ends at the same vertex is said to be *closed*; otherwise it is *open.* For example, when $r = 5$, the walk 3, 4, 4, 0, 1, 1, 1, 2 is an open 7-walk. It makes 4 forward moves and 3 stationary moves. Another way to describe this walk would be

$$X = (x_0; x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (3; F, S, F, F, S, S, F)$$

where x_0 indicates the initial vertex and the other values of x_i indicate whether the ith step is forward or stationary. Clearly, an n-walk that begins at x_0 and makes m forward moves will end up at vertex $x_0 + m \pmod r$.

Combinatorial Proof of Theorem 1

Question: How many closed n-walks on C_r begin at vertex 0?

Answer 1: Starting at vertex 0, there are $\binom{n}{m}$ n-walks that take m forward steps. For a walk to be closed, m must be a multiple of r. Consequently, our first answer is simply $\sum_{k\geq 0}\binom{n}{kr}$.

Answer 2: First we observe that there are as many closed n-walks that start at vertex 0 as start at vertex 1 or 2 or ... or $r-1$. Thus it suffices to prove that the total number of closed n-walks on C_r is $\sum_{j=0}^{r-1}(1+\omega^j)^n$. We accomplish this by assigning each n-walk (whether it be open or closed) a *weight* that will depend on its initial vertex and the number of forward moves. Specifically, an n-walk with initial vertex $x_0 = j$ that makes m forward moves will be assigned a weight of ω^{jm}. The 7-walk on C_5 in the previous example has $j = 3$, and $m = 4$ and therefore has weight $\omega^{12} = \omega^2$ since $\omega^5 = 1$. Equivalently, a walk that begins at vertex j and ends at vertex $j+m \pmod r$ has weight ω^{jm}. In particular, any closed walk will have weight $\omega^0 = 1$.

Another way to think of the weight of a walk beginning at vertex j is that each stationary step is given weight 1 and each forward step in the walk is given weight ω^j, and the weight of the walk is defined as the product of the weights of its steps. Consequently, the *total* weight of *all* n-walks that begin at j is $(1+\omega^j)^n$, since each $(1+\omega^j)$ represents a choice in our walk to make a stationary or forward move. (Alternatively, $(1+\omega^j)^n = \sum_{k\geq 0}\binom{n}{k}\omega^{jk}$ is the sum of the weights of all n-walks starting at j since $\binom{n}{k}\omega^{jk}$ is the total weight of all such walks with k forward steps.) Summing over all possible starting points,

$$\sum_{j=0}^{r-1}(1+\omega^j)^n \tag{5}$$

counts the total weight of all n-walks (open and closed) on C_r.

Our goal is to show that (5) counts the total number of all *closed* n-walks on C_r. Since each closed walk has weight 1, it suffices to show that the *total weight of all open walks is zero*. Consider an open walk X_0 that begins at vertex 0 and ends at vertex $m \neq 0$. Then X_0 generates the orbit $\{X_0, X_1, \ldots, X_{r-1}\}$ where walk X_j starts at vertex j, and then follows the same forward and stationary instructions as X_0, ending at vertex $j+m \pmod r$, with weight ω^{jm}. Summing a finite geometric series, the total weight of the n-walks in this orbit is

$$\sum_{j=0}^{r-1}\omega^{jm} = \frac{1-\omega^{mr}}{1-\omega^m} = 0,$$

since $\omega^r = 1$ and $\omega^m \neq 1$. Since every open walk appears in exactly one orbit, each with total weight zero, the total weight of all open walks is zero, as desired. Summarizing, for walks on C_r,

$$\begin{aligned}\text{the number of closed } n\text{-walks} &= \text{the total weight of all closed } n\text{-walks}\\ &= \text{the total weight of all } n\text{-walks}\\ &= \sum_{j=0}^{r-1}(1+\omega^j)^n.\end{aligned}$$

Hence, the number of closed n-walks that begin at 0 is $\frac{1}{r}\sum_{j=0}^{r-1}(1+\omega^j)^n$. ■

Combinatorial Proof of Theorem 2

In this proof, an n-walk on C_r that starts at vertex j and makes m forward moves is defined to have weight $\omega^{-ja}\omega^{mj} = \omega^{(m-a)j}$. Hence any walk that makes $a + rk$ forward moves has weight $\omega^{rkj} = 1$. Just like in the proof of Theorem 1, the total weight of all n-walks on C_r is $\sum_{j=0}^{r-1} \omega^{-ja}(1 + \omega^j)^n$. The theorem follows since the walks that make forward progress $m \neq a + rk$ can be placed into orbits of total weight $\sum_{j=0}^{r-1} \omega^{(m-a)j} = 0$. ■

Theorems 1 and 2 can also be expressed in terms of trigonometric functions [**1**], sometimes without mentioning any complex numbers. Suppose $\nu = e^{i\pi/r}$ is a primitive $2r$th root of unity so that $\nu^2 = \omega$. Then using Euler's formula, $e^{-i\theta} + e^{i\theta} = 2\cos\theta$, we may write the summand as

$$(1 + \omega^j)^n = [v^j(v^{-j} + v^j)]^n = v^{nj}(e^{-i\pi j/r} + e^{i\pi j/r})^n = v^{nj}(2\cos(\pi j/r))^n.$$

In particular, if n is a multiple of r, say $n = qr$, then

$$\sum_{k\geq 0}\binom{n}{rk} = \frac{2^n}{r}\sum_{j=0}^{r-1}(-1)^{qj}\left(\cos\frac{\pi j}{r}\right)^n \tag{6}$$

can be expressed entirely with real numbers. This is the form presented in [**1**]. Likewise, Theorem 2 simplifies to the same right hand side of (6) when $n = qr + 2a$.

Where do we go from here? A natural problem might be to try to count walks on other graphs to discover other identities. Conversely, we hope this technique may allow us to combinatorially understand other identities that mix binomial coefficients with complex numbers. For example, Identity 2.24 in [**1**] says for $r > 1$,

$$\sum_{k\geq 1}\frac{1}{\binom{kr}{r}} = \sum_{k=1}^{r-1} -\omega^k(1-\omega^k)^{r-1}\log\frac{1-\omega^k}{-\omega^k},$$

where ω is a primitive rth root of unity. Perhaps with the right combinatorial perspective, this identity will not appear nearly so complex after all.

Acknowledgment The authors thank Keith Dsouza and Michael Krebs, whose talk at the Fall 2009 meeting of the Southern California and Nevada Section of the MAA provided the motivation for this work.

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Summary Summary pending.