The BINGO Paradox

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Believe it or not, when a large number of people play bingo, it is much more likely that the winning card has a completed horizontal row than a completed vertical column. How can this be? If you randomly mark off numbers on your own bingo card, you are just as likely to get a horizontal bingo as a vertical bingo. Why should the *winning* card be any different?

A computer program was written that generated 1,000 random bingo cards and played the game 100,000 times. To our surprise, horizontal winners were almost twice as likely as a vertical winner.

To better understand this paradox, let's review the rules of this popular game. A typical bingo card, like the one in figure 1, has five columns, labeled B, I, N, G, and O, and each column has five numbers underneath it. Column B has five numbers from 1 through 15, in a random order. Similarly columns I, N, G, and O have five random numbers from 16 to 30, 31 to 45, 46 to 60, and 61 to 75, respectively. Some bingo cards have a free space in the middle of column N.

The caller calls out numbers—such as "B11!"—pulled randomly from a container. Players place markers on the corresponding spaces on their card if they have them. The first person to complete a row, column, or diagonal yells "bingo!" and wins a prize. In the analysis that follows, we will initially ignore the effect of the free space, but we will consider it later.

Suppose the first eight numbers were drawn in this order: B11, I23, G58, B13, I21, N34, G55, and O75. With these numbers, each letter has appeared, so it is possible for there to be a bingo card with a horizontal or diagonal win. But since no letter has appeared five times, it is impossible to have a vertical bingo at this point. This suggests a mathematical question that we can sink our teeth into. For a randomly generated sequence of bingo numbers, *what is the probability that all five letters appear before any single letter appears five times?*

B	I	N	G	O
3	27	34	50	67
6	19	35	48	65
7	25	FREE SPACE	54	61
13	26	36	55	64
10	30	42	49	70

Figure 1. A typical bingo card.

Analyzing the Paradox

For simplicity, let's assume we are playing with so many bingo cards that as soon as all five letters appear, we have a horizontal winner and as soon as one letter appears five times, we have a vertical winner.

We define a *horizontal sequence* to be an arrangement of the 75 bingo numbers so that all five letters appear before any letter appears five times. Otherwise, it is called a *vertical sequence*. We say a sequence has property H_n if it becomes a horizontal sequence on the nth number and has property V_n if it becomes a vertical sequence on the nth number. A sequence that begins with the eight numbers given above would have property H_8. Note that it is impossible for a sequence to be both horizontal and vertical since a number like O75 could not simultaneously be the first time that the letter O appears and the fifth time that the letter O appears.

The probability of a horizontal sequence in five draws (the minimum number possible) is

$$P(H_5) = \frac{60}{74}\cdot\frac{45}{73}\cdot\frac{30}{72}\cdot\frac{15}{71} \approx 0.04400,$$

since after the first number is drawn, 60 of the remaining 74 numbers will produce a new letter, then

45 of the remaining 73 numbers will produce a third letter, and so on. Similarly, the chance of a vertical sequence in five draws is

$$P(V_5) = \frac{14}{74} \cdot \frac{13}{73} \cdot \frac{12}{72} \cdot \frac{11}{71} \approx 0.00087.$$

So a horizontal sequence is about 50 times more likely than a vertical sequence on the fifth draw.

What about after the fifth draw? Here, the counting gets a little more complicated, but we can do it. Let's find the probability of achieving a horizontal sequence in exactly 10 draws. Note that there are 75! equally likely sequences of bingo numbers. How many of them result in a horizontal sequence on the 10th draw? Let's choose the 10th number first. There are 75 possibilities, so let's mentally choose the 10th to be O75.

The nine previous numbers can have four possible *shapes* based on how many of each of the letters B, I, N, and G appear: 4311, 4221, 3321, or 3222. For example, the sequence N31, N41, G59, I26, B5, N35, B8, B9, B7 (inspired by the digits of pi) has shape 4311, since a letter appears four times (B), another letter appears three times (N), and the other letters (I and G) each appear once.

The horizontal shapes are *partitions* of the integer 9 into four positive parts, where all parts have size at most 4. Thus, a shape like 5211 is not allowed since it contains five of the same letter and would therefore be a vertical sequence.

The letters B, I, N, and G can be given a shape of 4311 in 12 different ways (BBBBIIING, BBBBINNNG, and so on). Likewise, there are 12 ways to give them a shape of 4221, 12 ways to give them a shape of 3321, but only four ways to give them a shape of 3222 (we have four choices for which letter appears three times, and the other letters must all appear twice). In general, a four-digit shape can be assigned to B, I, N, and G in 1, 4, 6, 12, or 24 ways depending on whether it consists of all the same digit, a tripled digit, two pairs of digits, one pair of digits, or all different digits, respectively.

Once we have determined how many of each letter is to be used, we can count the ways to assign them numbers using binomial coefficients. Recall that the binomial coefficient

$$\binom{15}{k} = \frac{15!}{k!(15-k)!}$$

is the number of ways we can choose k items out of a collection of 15 items. For example, the number of ways to assign numbers to BBBBIIING is

$$\binom{15}{4}\binom{15}{3}\binom{15}{1}\binom{15}{1} = 455 \cdot 105 \cdot 15^2 = 10{,}749{,}375.$$

Say we choose the numbers B1, B2, B3, B4, I16, I17, I18, N31, G46. Then these nine numbers can be arranged, like Scrabble tiles in a rack, in 9! ways. Finally, the remaining 65 numbers (appearing after O75) can be arranged in 65! ways. Putting this all together, the probability that a bingo sequence becomes horizontal on the 10th draw with shape 4311 is

$$P(4311) = \binom{15}{4}\binom{15}{3}\binom{15}{1}\binom{15}{1}\frac{75 \cdot 12 \cdot 9!65!}{75!} \approx 0.01517.$$

Similarly, the probabilities of becoming horizontal with the other possible shapes are

$$P(4221) = \binom{15}{4}\binom{15}{2}\binom{15}{2}\binom{15}{1}\frac{75 \cdot 12 \cdot 9!65!}{75!} \approx 0.02451,$$

$$P(3321) = \binom{15}{3}\binom{15}{3}\binom{15}{2}\binom{15}{1}\frac{75 \cdot 12 \cdot 9!65!}{75!} \approx 0.03540,$$

and

$$P(3222) = \binom{15}{3}\binom{15}{2}\binom{15}{2}\binom{15}{2}\frac{75 \cdot 4 \cdot 9!65!}{75!} \approx 0.01906.$$

Altogether, the probability of achieving a horizontal sequence on the 10th draw is

$$\begin{aligned} P(H_{10}) &= P(4311) + P(4221) + P(3321) + P(3222) \\ &\approx 0.09415. \end{aligned}$$

We perform similar calculations to find $P(V_{10})$, the probability of a vertical sequence on the 10th draw. Again, there are 75 possibilities for the 10th draw, which we will assume is O75. The previous nine numbers must include exactly four Os, which can be chosen in $\binom{14}{4}$ ways. The remaining five letters can have one of four possible shapes: 4100, 3200, 3110, or 2210. We exclude the shapes 5000 and 2111, since the first shape would create an earlier vertical sequence and the second shape (combined with the other Os) would create an earlier horizontal sequence. Each of the shapes has one digit that appears twice and can therefore be assigned letters in 12 ways. Therefore,

$$P(4100) = \binom{14}{4}\binom{15}{4}\binom{15}{1}\frac{75 \cdot 12 \cdot 9!65!}{75!} \approx 0.00223,$$

$$P(3200) = \binom{14}{4}\binom{15}{3}\binom{15}{2}\frac{75 \cdot 12 \cdot 9!65!}{75!} \approx 0.00519,$$

n	Shapes	$P(H_n)$	$P(V_n)$	Ratio	Sum	Cumulative
5	1	0.04400	0.00087	50.57	0.0449	0.0449
6	1	0.08800	0.00373	23.60	0.0917	0.1366
7	2	0.11350	0.00956	11.87	0.1231	0.2597
8	3	0.12052	0.01903	6.33	0.1396	0.3992
9	4	0.11220	0.02902	3.87	0.1412	0.5404
10	4	0.09415	0.03652	2.58	0.1307	0.6711
11	5	0.07191	0.03968	1.81	0.1116	0.7827
12	4	0.04972	0.03748	1.33	0.0872	0.8699
13	4	0.03075	0.03085	1.00	0.0616	0.9315
14	3	0.01658	0.02178	0.76	0.0384	0.9698
15	2	0.00742	0.01260	0.59	0.0200	0.9898
16	1	0.00254	0.00558	0.45	0.0081	0.9980
17	1	0.00052	0.00151	0.34	0.0020	1.0000
Total	**35**	**0.752**	**0.248**		**1.000**	

Table 1. The probabilities of a vertical or a horizontal bingo.

$$P(3110) = \binom{14}{4}\binom{15}{3}\binom{15}{1}\binom{15}{1}\frac{75 \cdot 12 \cdot 9!65!}{75!} \approx 0.01113,$$

and

$$P(2210) = \binom{14}{4}\binom{15}{2}\binom{15}{2}\binom{15}{1}\frac{75 \cdot 12 \cdot 9!65!}{75!} \approx 0.01797.$$

So, the probability of achieving a vertical sequence on the 10th draw is

$$P(V_{10}) = P(4100) + P(3200) + P(3110) + P(2210) \approx 0.03652.$$

Comparing $P(V_{10})$ to $P(H_{10})$, we see that even on the 10th draw, horizontal sequences are more than twice as likely to appear as vertical sequences.

Every sequence will become horizontal or vertical within 17 draws: After 16 draws, we can have a sequence of four Bs, Is, Ns, and Gs, say, but the next number will be either an O (creating a horizontal sequence) or a B, I, N, or G (creating a vertical sequence). We summarize our findings in table 1.

The upshot is that the probability of a horizontal sequence is 75.2 percent, which is about three times more likely than a vertical sequence. A sequence becomes horizontal or vertical by the 12th draw about 87 percent of the time. In all these cases, horizontal sequences are much more likely than vertical sequences. When it happens on the 13th draw (about 6 percent of the time) the sequences have almost the same probability, and when it happens after the 13th draw, which is only about 7 percent of the time, then the vertical sequences have the edge.

When all cards begin with a free space in the middle, the chance of a vertical sequence increases slightly, since column N now has 15 numbers instead of 14 numbers to cover the remaining four spaces. Joe Kisenwether and Dick Hess independently discovered that when the free space is used, the chance of a horizontal win is 73.73 percent (see Dick Hess's *The Population Explosion and Other Mathematical Puzzles*, World Scientific, 2016).

The Numbers of Shapes

Although we have answered our original question, more interesting mathematics is lurking behind the analysis.

When we enumerated sequences with properties H_{10} and V_{10}, we had to analyze—in both cases—exactly four shapes. This is not a coincidence. For each n, the sequences that are horizontal and vertical on the nth draw yield the same number of shapes. This is not obvious. The number of shapes for H_n is the number of partitions of $n-1$ by four positive integers less than 5, and the number of shapes for V_n is the number of partitions of $n-5$ by four nonnegative integers less than 5, at least one of which is 0. What's going on?

We will illustrate the one-to-one correspondence between the shapes for H_n and the shapes for V_n in the case $n = 10$, but the reasoning is the same for any n.

The shapes used for the enumeration of H_{10} consist of four positive numbers that add to 9, where all numbers are less than or equal to 4. These partitions, 4311, 4221, 3321, and 3222, are displayed pictorially in figure 2. Such representations are called *Ferrers diagrams*. Note that they fit in a 4-by-4 box.

If we subtract 1 from each value, or equivalently, delete the first columns of dots in the Ferrers diagrams,

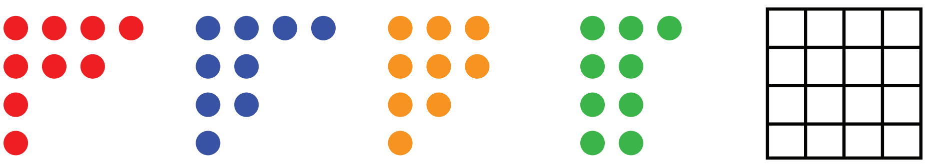

Figure 2. Ferrers diagrams for the partitions 4311, 4221, 3321, and 3222 (left to right) fit into a 4-by-4 box.

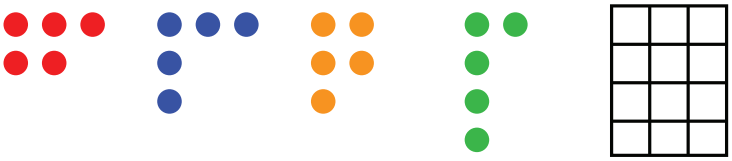

Figure 3. Partitions of 5 into nonnegative values less than or equal to 3 fit into a 4-by-3 box.

Figure 4. The conjugate partitions of those in figure 3 fit into a 3-by-4 box.

as in figure 3, we get partitions of 5 into four nonnegative parts, where all numbers are less than or equal to 3. The partitions now fit inside a 4-by-3 box.

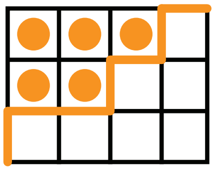

Figure 5. The lattice path corresponding to the partition 3200.

Next, interchange the rows and columns of dots to create the *conjugate partitions* in figure 4. These Ferrers diagrams fit into a 3-by-4 box, or equivalently a 4-by-4 box with an empty last row. They correspond to partitions of 5 by four nonnegative integers less than or equal to 4, at least one of which is 0. In particular, these are the V_{10} shapes 2210, 3110, 3200, and 4100.

Thus, this technique gives a bijection between the shapes for H_{10} and the shapes for V_{10}; and the same technique works for other values of n.

In fact, we can get more information from these diagrams. In the V_n case, each partition carves out a *lattice path* from the point (0,0) to (4,3) in the 3-by-4 box. For example, the partition 3200 creates the lattice path in figure 5. Consequently, the number of shapes needed to compute all of the vertical sequences (and hence the number needed to compute all of the horizontal sequences) corresponds to the number of lattice paths from (0,0) to (4,3). Since each lattice path takes, in some order, four steps to the right and three steps up, the total number of shapes is

$$\binom{7}{3} = \frac{7!}{3!4!} = 35,$$

as seen at the bottom of the second column of table 1.

Finally, there is a slick way to generate the number of shapes for each n (the rest of the entries in the second column of table 1) using *q-binomial coefficients,* which are polynomial generalizations of binomial coefficients.

First we replace the integer m with the polynomial

$$m_q = 1 + q + q^2 + \cdots + q^{m-1} = \frac{1-q^m}{1-q}.$$

For example,

$$7_q = 1 + q + q^2 + \cdots + q^6 = \frac{1-q^7}{1-q}.$$

Then a binomial coefficient like

$$\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$$

has a corresponding q-binomial coefficient

$$\binom{7}{3}_q = \frac{7_q \cdot 6_q \cdot 5_q}{3_q \cdot 2_q \cdot 1_q} = \frac{(1-q^7)(1-q^6)(1-q^5)}{(1-q^3)(1-q^2)(1-q^1)}$$
$$= \frac{1 - q^5 - q^6 - q^7 + q^{11} + q^{12} + q^{13} - q^{18}}{1 - q - q^2 + q^4 + q^5 - q^6}.$$

Believe it or not, after simplifying this rational function, we obtain a 12th-degree polynomial in which the coefficient of the q^n term is the number of partitions of the integer n that fit in a 4-by-3 box (see *Integer Partitions* by George E. Andrews and Kimmo Ericsson, Cambridge University Press, 2004, for a justification). In other words, it's the shape-counting polynomial

$$1 + q + 2q^2 + 3q^3 + 4q^4 + 4q^5 +$$
$$5q^6 + 4q^7 + 4q^8 + 3q^9 + 2q^{10} + q^{11} + q^{12},$$

as seen in the second column of table 1. ■

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