# THE LUCAS TRIANGLE RECOUNTED

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## 1. Introduction

In [2], Neville Robbins explores many properties of the *Lucas triangle*, an infinite triangular array with properties similar to Pascal's triangle. In this paper, we provide a combinatorial explanation for the entries of this triangle. This interpretation results in extremely quick and intuitive proofs of most of the properties (proved mostly by induction in [2]) and allows for a natural generalization, with equally transparent proofs.

Using Robbins' notation, we define the number $\begin{bmatrix} n \\ k \end{bmatrix}$ by the initial conditions

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 1 \quad \text{for } n \geq 1, \qquad \begin{bmatrix} n \\ n \end{bmatrix} = 2 \quad \text{for } n \geq 0,$$

and the Pascal-like recurrence for $1 \leq k \leq n-1$,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

For combinatorial convenience, we shall define $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ whenever $n < 0$ or $k < 0$ or $k > n$. This allows the recurrence to be true for all values of $n$ and $k$ except for $n = k = 0$ and $n = 1$, $k = 0$.

Below we list rows 0 through 9 of the Lucas triangle:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | | | | | | | | | |
| 1 | 2 | | | | | | | | |
| 1 | 3 | 2 | | | | | | | |
| 1 | 4 | 5 | 2 | | | | | | |
| 1 | 5 | 9 | 7 | 2 | | | | | |
| 1 | 6 | 14 | 16 | 9 | 2 | | | | |
| 1 | 7 | 20 | 30 | 25 | 11 | 2 | | | |
| 1 | 8 | 27 | 50 | 55 | 36 | 13 | 2 | | |
| 1 | 9 | 35 | 77 | 105 | 91 | 49 | 15 | 2 | |
| 1 | 10 | 44 | 112 | 182 | 196 | 140 | 64 | 17 | 2 |

## 2. Combinatorial Interpretation

The connection to Lucas numbers is easy to see as follows. As is well-known [1], $L_n$ counts the ways to tile a bracelet of length $n$ with squares and dominoes. The *cells* of the bracelet are labeled from 1 through $n$ and we define the *first tile* to be the tile that covers cell 1, the cell that is to the right of *the clasp*. For example, the Lucas number $L_4 = 7$, reflects the fact that there are seven bracelets of length four, namely $dd$ (two ways), $dss$ (two ways), $sds$, $ssd$, and $ssss$ where $d$ denotes a domino, $s$ denotes a square; the first two tilings have two representations, depending on whether the initial domino covers the clasp (cells 4 and 1) or covers cells 1 and 2.

**Theorem 1.** *For $n \geq 1$, $\left[\begin{matrix} n \\ k \end{matrix}\right]$ counts bracelets containing $n$ tiles with exactly $k$ dominoes (and therefore $n - k$ squares).*

*Proof.* Let $B(n, k)$ denote the number of bracelets with $k$ dominoes and $n - k$ squares. We prove that $B(n, k) = \left[\begin{matrix} n \\ k \end{matrix}\right]$ by showing that $B(n, k)$ satisfies the same initial conditions and recurrence. Clearly, there is one bracelet with $n$ squares and no dominoes and there are two bracelets with $n$ dominoes and no squares (depending on the location of the initial domino). Thus $B(n,0) = 1 = \begin{bmatrix} n \\ 0 \end{bmatrix}$ and $B(n,n) = 2 = \begin{bmatrix} n \\ n \end{bmatrix}$. For the recurrence, notice that a bracelet with $k$ dominoes and $n-k$ squares will either end with a square (preceded by $k$ dominoes and $n-1-k$ squares) or end with domino (preceded by $k-1$ dominoes and $n-k$ squares). That is, $B(n,k) = B(n-1,k) + B(n-1,k-1)$ satisfies the same recurrence as $\begin{bmatrix} n \\ k \end{bmatrix}$, and therefore $B(n,k) = \begin{bmatrix} n \\ k \end{bmatrix}$. □

Using this combinatorial interpretation, most of the identities presented in [2] can be proved by inspection (more precisely, by inspecting the size of a set in two different ways). We note that a bracelet with $n$ tiles consisting of $k$ dominoes and $n-k$ squares will have length $n+k$.

It follows that for $n \geq 1$,

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = n+1$$

since a bracelet with $n$ tiles with just one domino has length $n+1$ and that domino can start on any of the $n+1$ cells. Likewise, for $n \geq 1$,

$$\begin{bmatrix} n \\ n-1 \end{bmatrix} = 2n-1$$

since a bracelet with $n-1$ dominoes and one square has length $2n-1$ and therefore has $2n-1$ places to put its lone square. Similarly, for $n \geq 2$,

$$\begin{bmatrix} n \\ n-2 \end{bmatrix} = (n-1)^2$$

since a bracelet of length $2n-2$ with two squares can be obtained by inserting the squares in any two cells of opposite parity.

The following sums are also easy to "see." For $n \geq 1$,

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = 3(2^{n-1})$$

since the left side counts bracelets with $n$ tiles with an arbitrary number of dominoes. There are three choices for the first tile (one possible square and two possible dominoes) followed by two choices for each of the following $n-1$ tiles. By the same reasoning, we have, for $n \geq 2$,

$$\sum_{k \geq 0} \begin{bmatrix} n \\ 2k \end{bmatrix} = \sum_{k \geq 0} \begin{bmatrix} n \\ 2k+1 \end{bmatrix} = 3(2^{n-2})$$

since if we specify the parity of the number of dominoes, then the last tile is forced. Finally notice that $\begin{bmatrix} n-i \\ i \end{bmatrix}$ counts bracelets of length $n$ with $i$ dominoes and therefore

$$\sum_{i \geq 0} \begin{bmatrix} n-i \\ i \end{bmatrix} = L_n.$$

Notice that when $p$ is prime, we have for all $1 \leq i \leq (p-1)/2$,

$$p \,\middle|\, \begin{bmatrix} p-i \\ i \end{bmatrix}$$

since every bracelet of prime length $p$ with at least 1 domino has $p$ distinct bracelets in its "orbit," obtained by shifting each tile clockwise $k$ units as $k$ varies from 0 to $p-1$. The bracelets are distinct since $p$ is prime. Even Robbins's following extension can be appreciated combinatorially.

**Proposition:** Let $p$ be an odd prime, then for all $j$ such that $1 \leq j \leq p-2$ and for all $i$ such that $j+1 \leq i \leq p-1$, we have $p \,\Big|\, \begin{bmatrix} p+j \\ i \end{bmatrix}$.

*Proof.* Let $X$ be a bracelet with $p+j$ tiles, where $1 \leq j \leq p-2$, with $i$ dominoes where $j+1 \leq i \leq p-1$. Here we obtain the orbit of $X$ by keeping the *first* $j$ tiles fixed, and then rotating the remaining $p$ tiles one *tile* at a time. Notice that the bounds on $i$ and $j$ ensure that the fixed part has at least one tile, and that the rotating part has at least one square and at least one domino. (If the rotating part of $X$ were all squares or all dominoes, then its orbit would have size one or two, respectively.) Thus, since $p$ is prime, every orbit of $X$ has $p$ elements, as desired. □

Alternating sums can also be easily handled combinatorially. For example, for $n \geq 2$, the identity

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} = 0$$

says that, among bracelets with $n$ tiles, there are as many bracelets with an even number of dominoes as with an odd number of dominoes. This is easy to see by the simplest of involutions: toggling the last tile. In other words, if the last tile of your bracelet is a square then turn it into a domino; if the last tile is a domino, then turn it into a square. (Note that the condition that $n \geq 2$ assures that the last tile is not the same as the first tile.) Either way, the parity of the number of dominoes has changed. In this way, every bracelet with $n$ tiles and an even number of dominoes "holds hands with" a bracelet with $n$ tiles and an odd number of dominoes.

The preceding identity and argument can be generalized to give:

$$\sum_{k=0}^{m} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} = (-1)^m \begin{bmatrix} n-1 \\ m \end{bmatrix}.$$

For this identity, we apply the same toggling argument as before, but now we are restricted to bracelets with $n$ tiles that have at most $m$ dominoes. The only bracelets that are not matched up are those that have $m$ dominoes and end with a square. (These bracelets are unmatched because toggling the last square would create a bracelet with $m+1$ dominoes, which exceeds our upper bound.) Since there are $\begin{bmatrix} n-1 \\ m \end{bmatrix}$ bracelets of this type, all of which have $m$ tiles, the identity follows.

Next, we note that entries of the Lucas triangle can be expressed in terms of the classical binomial coefficients that inhabit Pascal's triangle. Specifically, for $n \geq 2$ and $k \geq 0$,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k} + \binom{n-1}{k-1}.$$

For any bracelet with $n$ tiles and $k$ dominoes, it either has a domino covering the clasp (resulting in an *out-of-phase* bracelet) or it does not (resulting in an *in-phase* bracelet.) The first binomial coefficient counts the in-phase bracelets, since we simply choose which $k$ of the $n$ tiles are dominoes. The number of out-of-phase bracelets is $\binom{n-1}{k-1}$ since the first tile must be a domino covering the clasp, and then we can freely choose $k-1$ of the remaining $n-1$ tiles to be dominoes.

In fact $\begin{bmatrix} n \\ k \end{bmatrix}$ can be expressed even more directly in terms of binomial coefficients, namely

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n+k}{n}\binom{n}{k},$$

or equivalently,

$$n\begin{bmatrix} n \\ k \end{bmatrix} = (n+k)\binom{n}{k}.$$

The left side counts the ways to create a bracelet with $n$ *tiles,* containing $k$ dominoes, and placing a star on one of the $n$ tiles. (If the selected tile is a domino, we shall place the star on its left half.) The right side counts in-phase bracelets with $n$ tiles, containing $k$ dominoes where we place a star on one of its

$n+k$ *cells.* We show that these two sets have the same size by creating a bijection between the two sets. Let $X$ be a bracelet from the first set. Then $X$ has length $n+k$ and has a star on a tile that begins on some cell $j$, where $1 \leq j \leq n+k$. If $X$ is in-phase, then we leave the bracelet unchanged, and simply move the star from the tile to the cell $j$. If $X$ is out-of-phase, then we rotate $X$ counterclockise $j-1$ units so that it becomes an in-phase bracelet beginning with the tile that originally had the star. Then we transfer the star to the cell below the right half of the domino that used to cover the clasp. (This would put the star on cell $n+k+2-j$.) Since this process is easily reversed by examining the tile that covers the cell with the star, we have our desired bijection.

The next two identities make use of the *complement* of a tiling. Given a tiling $X$, we obtain its complement $X^*$ by toggling each tile. For example, if $X = ssdds$ then $X^* = ddssd$. Recalling that $F_m$ counts the set of tilings (or in-phase bracelets) of length $m-1$, we obtain the next identity (misstated in [2]). For $n \geq 1$,

$$F_{2n+2} = \sum_{i=0}^{n} \begin{bmatrix} n+i \\ 2i \end{bmatrix}.$$

To see this, let $X$ be a tiling of length $2n+1$. If $X$ begins with a square, then we create an in-phase bracelet $Y$ by deleting the first square, then taking the complement of the rest of $X$. If we let $X_1$ be the tiling $X$ with its first tile missing, then our mapping can be denoted by

$$sX_1 \rightarrow X_1^* \quad \text{(in-phase)}.$$

Notice that since $X$ has odd length, then the number of squares in $X$ must be an odd number $2i+1$ for some $0 \leq i \leq n$. It follows that $X$ has $n+i+1$ tiles comprising $2i+1$ squares and $n-i$ dominoes. Hence the bracelet $Y$ has $n+i$ tiles with $2i$ dominoes, and all in-phase bracelets of this type can be obtained this way. On the other hand, if $X$ begins with a domino, then we obtain the out-of-phase bracelets by the mapping

$$dX_1 \to dX_1^* \quad \text{(out-of-phase)}$$

where the initial domino of $X$ covers the clasp of $Y$, and all subsequent tiles of $X$ are replaced by their complement. Here, if $X$ has $2i-1$ squares for some $1 \leq i \leq n$, then $X$ will contain $n+i$ tiles comprising $2i-1$ squares and $n+1-i$ dominoes, and therefore $Y$ will contain $n+i$ tiles with exactly $2i$ dominoes as desired. We leave it to the reader to verify that the exact same mapping leads to the identity

$$F_{2n+1} = \sum_{i=0}^{n-1} \left[ \begin{matrix} n+i \\ 2i+1 \end{matrix} \right].$$

## 3. The Gibonacci Triangle

The Lucas numbers are a special case of the *Gibonacci numbers* $G_n$ defined by arbitrary initial conditions $G_0$ and $G_1$, and for $n \geq 2$,

$$G_n = G_{n-1} + G_{n-2}.$$

It is easy to show (as done in [1]) that $G_n$ has the following combinatorial interpretation. When $G_0$ and $G_1$ are nonnegative integers and $n \geq 1$, $G_n$ counts *phased tilings* of length $n$. These are just like traditional tilings with squares and dominoes, but the first tile is given a *phase.* If the initial tile is a domino, then it can be assigned one of $G_0$ phases; if the initial tile is a square, then it can be assigned one of $G_1$ phases. Notice that when $G_0 = 2$ and $G_1 = 1$, that $G_n = L_n$

and the $G_0 = 2$ phases of an initial domino indicate whether or not an initial domino covers the clasp. When $G_0 = 0$ and $G_1 = 1$, then $G_n = F_n$ and we are counting length $n$ tilings that must begin with a square (since there are no initial dominoes) and therefore $G_n$ counts arbitrary tilings of length $n-1$. Although this definition requires $G_0$ and $G_1$ to be nonnegative integers, a similar interpretation can be given for arbitrary real or complex numbers (as shown in [1]) but we will not pursue that here.

We define the number $\begin{bmatrix} n \\ k \end{bmatrix}_G$ by the initial conditions

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_G = G_1 \text{ for } n \geq 1, \quad \begin{bmatrix} n \\ n \end{bmatrix}_G = G_0 \quad \text{for } n \geq 0,$$

and for $1 \leq k \leq n-1$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_G = \begin{bmatrix} n-1 \\ k \end{bmatrix}_G + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_G.$$

Also we let $\begin{bmatrix} n \\ k \end{bmatrix}_G = 0$ when $n < 0$ or $k < 0$ or $k > n$.

For example, the initial terms $G_0 = 4$ and $G_1 = 9$, generate the Gibonacci sequence: $4, 9, 13, 22, 35, 57, 92, 149, 241, 390, \ldots$ The first ten rows of the Gibonacci Triangle are:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 9 | | | | | | | | | |
| 4 | 9 | | | | | | | | |
| 4 | 13 | 9 | | | | | | | |
| 4 | 17 | 22 | 9 | | | | | | |
| 4 | 21 | 39 | 31 | 9 | | | | | |
| 4 | 25 | 60 | 70 | 40 | 9 | | | | |
| 4 | 29 | 85 | 130 | 110 | 49 | 9 | | | |
| 4 | 33 | 114 | 215 | 240 | 159 | 58 | 9 | | |
| 4 | 37 | 147 | 329 | 455 | 399 | 217 | 67 | 9 | |
| 4 | 41 | 184 | 476 | 784 | 854 | 616 | 284 | 76 | 9 |

Arguing exactly as before, we have a simple combinatorial interpretation of $\begin{bmatrix} n \\ k \end{bmatrix}_G$.

**Theorem 2.** *For $n \geq 1$, $k \geq 0$, $\begin{bmatrix} n \\ k \end{bmatrix}_G$ counts phased tilings with $n$ tiles containing exactly $k$ dominoes (and therefore $n-k$ squares).*

This combinatorial interpretation allows us to immediately generalize most of the preceding theorems. (It also immediately gives us a generating function, namely $\begin{bmatrix} n \\ k \end{bmatrix}_G$ is the $a^k b^{n-k}$ coefficient of $(G_0 a + G_1 b)(a+b)^{n-1}$.) Their proofs are almost exactly the same as before, so we leave most of their details to the reader.

In our first set of identities, we decide on the length and phase of the initial tile, then proceed to choose the remaining $n-1$ (unphased) tiles. For $n \geq 1$,

$$\begin{bmatrix} n \\ 1 \end{bmatrix}_G = G_0 + G_1(n-1) = G_1 n + (G_0 - G_1),$$

$$\begin{bmatrix} n \\ n-1 \end{bmatrix}_G = G_1 + G_0(n-1) = G_0 n + (G_1 - G_0),$$

and for $n \geq 2$,

$$\begin{bmatrix} n \\ n-2 \end{bmatrix}_G = G_0 \binom{n-1}{2} + G_1(n-1),$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_G = (G_0 + G_1)2^{n-1} = G_2 2^{n-1},$$

$$\sum_{k \geq 0} \begin{bmatrix} n \\ 2k \end{bmatrix}_G = \sum_{k \geq 0} \begin{bmatrix} n \\ 2k+1 \end{bmatrix}_G = G_2 2^{n-2},$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_G = G_1 \binom{n-1}{k} + G_0 \binom{n-1}{k-1} = G_1 \binom{n}{k} + (G_0 - G_1)\binom{n-1}{k-1}.$$

Some identities generalize with virtually no changes to their statement or logic. For example, the sum of the diagonal entries of the Gibonacci triangle yields

Gibonacci numbers. For $n \geq 0$,

$$\sum_{i\geq 0} \begin{bmatrix} n-i \\ i \end{bmatrix}_G = G_n.$$

Although the prime identity

$$p \left| \begin{bmatrix} p-i \\ i \end{bmatrix} \right.$$

does not have a phased analog (because we cannot rotate the phased initial tile), the extension of that result works fine because the first $j$ tiles are fixed. Specifically, for any odd prime $p$ and for any $1 \leq j \leq p-2$ and $j+1 \leq i \leq p-1$,

$$p \left| \begin{bmatrix} p+j \\ i \end{bmatrix}_G \right. .$$

Likewise, the alternating identities, obtained by toggling the last tile, are completely unaffected by the initial tile. Thus, for $n \geq 2$,

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_G = 0,$$

$$\sum_{k=0}^{m} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_G = (-1)^m \begin{bmatrix} n-1 \\ m \end{bmatrix}_G .$$

## References

[1] A. T. Benjamin and J. J. Quinn, Proofs That Really Count: The Art of Combinatorial Proof, Mathematical Association of America, Washington, DC, 2003.
[2] N. Robbins, "The Lucas Triangle Revisited." *The Fibonacci Quarterly,* Vol. 43 (2005): pp. 142–148.

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