UNIFIED TILING PROOFS OF A FAMILY OF FIBONACCI IDENTITIES

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Abstract. In a recent work, Baxter and Pudwell mentioned the following identity for the Fibonacci numbers F_n and noted that it can be proven via induction: For all $n \geq 1$,

$$F_{2n} = 1 \cdot F_{2n-2} + 2 \cdot F_{2n-4} + \cdots + (n-1) \cdot F_2 + n.$$

We give a combinatorial proof of this identity and a companion identity. This leads to an infinite family of identities, which are also given combinatorial proofs.

1. Introduction

In a recent paper, Baxter and Pudwell [1] featured the following identity. For $n \geq 1$,

$$F_{2n} = F_{2n-2} + 2F_{2n-4} + \cdots + (n-1)F_2 + n. \tag{1}$$

The authors note that (1) can be proven by induction. Our initial goal is to prove (1) combinatorially with *tilings* and then extend our arguments to prove related identities. Recall (as in [2]) that $f_n = F_{n+1}$ counts the ways to tile a $1 \times n$ board using 1×1 squares and 1×2 dominoes. With this in mind, we rewrite (1) as follows. For $n \geq 1$,

$$f_{2n-1} = f_{2n-3} + 2f_{2n-5} + \cdots + (n-1)f_1 + n. \tag{2}$$

Once we prove (2) combinatorially, it is easy to identify the following "companion" identity: For $n \geq 1$,

$$f_{2n} = f_{2n-2} + 2f_{2n-4} + \cdots + nf_0 + 1. \tag{3}$$

In the next section, we provide tiling proofs of (2) and (3) and then generalize our results to a pair of infinite families of Fibonacci identities in a natural and unified manner. In the closing section, we highlight further generalizations of these results.

2. Tiling Proofs

We begin by providing a detailed proof of (2), which will motivate many of the proofs in the remainder of this note.

Proof. (of (2)) We identify a set of tilings that is counted in different ways by each side of the equality. The left side of (2) clearly counts the number of tilings of a $1 \times (2n-1)$ board.

For the right side, we focus on the location of the **second** square in a given tiling of a $1 \times (2n-1)$ board. Note that the possible locations of the second square in such a tiling are the cells 2, 4, 6, …, $2n-2$. Now, let $2j$ be the location of the second square in the tiling. To the right of this cell, the remaining spaces are tiled in $f_{2n-1-2j}$ ways (by definition). To the left of position $2j$, we have exactly one square and $j-1$ dominoes, which can be arranged in j ways. Thus, the number of such tilings is $jf_{2n-1-2j}$. Summing over all possible values of j gives us **most** of the right side of (2), namely

$$f_{2n-3} + 2f_{2n-5} + \cdots + (n-1)f_1.$$

Finally, the "non–homogeneous" term of n in the right side of (2) counts the tilings consisting of a single square, located in an odd numbered cell with $n-1$ dominoes. □

A similar argument establishes (3). The only difference is that there is only one tiling of length $2n$ that does not contain a second square, namely the all-domino tiling.

The previous argument can be generalized in multiple directions. We begin by noting that there is nothing special about focusing on the *second* square in a given tiling; we could focus on the location of the first square, the third square, and so on. With this in mind, we now count tilings of a board with a focus on the location of the pth square in the tiling where $p \geq 1$. This leads us to the following natural generalization of (2) and (3), which can be expressed in a single theorem:

Theorem 2.1. *For $p \geq 1$ and $m \geq 1$,*

$$f_m = \sum_{k=p}^{m} \binom{(k+p-2)/2}{p-1} f_{m-k} + \sum_{t=0}^{p-1} \binom{(m+t)/2}{t},$$

where we define the binomial coefficient $\binom{n}{r}$ to be zero when n is not an integer. In particular, the nonzero summands of the first summation are those where k has the same parity as p, and the nonzero summands of the second summation are those where t has the same parity as m. Notice that when $p = 2$ and $m = 2n - 1$ or $2n$, then we obtain the previous two identities.

Proof. (of Theorem 2.1) Like before, the left side counts tilings of length m. We argue that the right side also counts such tilings, by considering the location of the pth square, if it exists. Suppose the pth square is located at cell k for some k between p and m. To the right of that square, the board can be tiled in exactly f_{m-k} ways. Before that, we have exactly $p-1$ squares and $(k-p)/2$ dominos, provided that k and p have the same parity (otherwise no such tilings exist). Altogether we have $(p-1)+(k-p)/2 = (k+p-2)/2$ tiles, which can be arranged in $\binom{(k+p-2)/2}{p-1}$ ways, as desired. The second summand counts tilings with t squares where $t \leq p-1$. Such tilings have $(m-t)/2$ dominos, provided that t and m have the same parity, and those $(m+t)/2$ tiles can be arranged in $\binom{(m+t)/2}{t}$ ways, and the proof is complete. □

If we enumerate our tilings based on the location of the pth domino, then we obtain an even simpler expression, because we do not have parity issues to navigate, and we obtain the following result.

Theorem 2.2. *For $p \geq 1$ and $m \geq 1$,*

$$f_m = \sum_{k=2p}^{m} \binom{k-p-1}{p-1} f_{m-k} + \sum_{t=0}^{p-1} \binom{m-t}{t}.$$

Proof. The left side of the equation counts the number of ways to tile a $1 \times m$ board. Now, consider such a tiling and suppose the pth domino (if it exists) covers cells $k-1$ and k, where $k \geq 2p$. Then, as before there are f_{m-k} ways to tile the cells to the right of cell k. Prior to cell k, we arrange $p-1$ dominos and $k-2p$ squares, which can be done in $\binom{k-p-1}{p-1}$ ways, which explains the first summation. The second summation counts those tilings with $t \leq p-1$ dominos and $m-2t$ squares, which can be done in $\binom{m-t}{t}$ ways, as desired. □

3. Closing Thoughts

We close with two sets of thoughts on ways in which these ideas can be extended.

First, results of a form similar to those in Theorems 2.1–2.2 can be obtained for the Lucas numbers by considering tiling circular boards rather than linear boards. We leave the details to the reader.

Secondly, we can easily generalize Theorems 2.1–2.2 by allowing a different colors of square tiles and b different colors of domino tiles to be used in our tilings. In this context, we define a two-parameter family of sequences, which we will denote by u_n as follows:

$$u_n = au_{n-1} + bu_{n-2},$$

where $u_0 = 1$ and $u_1 = a$. In the colored tiling interpretation, we think of a and b as positive integers, but using a weighted tiling approach, a and b can be negative numbers, complex numbers, or polynomials. Theorems 2.1–2.2 can be easily generalized by keeping track of how many squares and dominoes appear on each side of the pth object in question. We then have the following new theorems:

Theorem 3.1. *For $p \geq 1$ and $m \geq 1$,*

$$u_m = \sum_{k=p}^{m} a^p b^{(k-p)/2} \binom{(k+p-2)/2}{p-1} u_{m-k} + \sum_{t=0}^{p-1} a^t b^{(m-t)/2} \binom{(m+t)/2}{t}.$$

Theorem 3.2. *For $p \geq 1$ and $m \geq 1$,*

$$u_m = \sum_{k=2p}^{m} a^{k-2p} b^p \binom{k-p-1}{p-1} u_{m-k} + \sum_{t=0}^{p-1} a^{m-2t} b^t \binom{m-t}{t}.$$

It is difficult to imagine discovering and proving Theorems 3.1–3.2 without the insights developed in this note.

References

[1] A. M. Baxter and L. K. Pudwell, *Ascent sequences avoiding pairs of patterns*, The Electronic Journal of Combinatorics, **22** no. 1, (2015).

[2] A. T. Benjamin and J. J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, Mathematical Association of America, Washington, DC, 2003.

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