A TILING INTERPRETATION OF A GENERALIZED ZECKENDORF'S THEOREM

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Abstract

Zeckendorf's theorem states that every positive integer can be decomposed uniquely into a sum of non-consecutive Fibonacci numbers. Previous works by Grabner and Tichy (1990) and Miller and Wang (2012) have found a generalization of Zeckendorf's theorem to a larger class of recurrent sequences, called *Positive Linear Recurrence Sequences* (PLRS's). We apply well-known tiling interpretations of recurrence sequences from Benjamin and Quinn (2003) to PLRS's. We exploit that tiling interpretation to create a new tiling interpretation specific to PLRS's that captures the behavior of this generalized Zeckendorf's theorem.

1. Introduction

In this paper, we use the combinatorial Fibonacci numbers defined by the recurrence relation $f_{n+1} = f_n + f_{n-1}$, with initial conditions $f_1 = 1$ and $f_2 = 2$. This generates the Fibonacci sequence $1, 2, 3, 5, 8, 13, 21, 34, \ldots$. Zeckendorf's theorem states that every positive integer can be decomposed uniquely into a sum of non-consecutive Fibonacci numbers.

There are many generalizations of the Fibonacci numbers, which involve changing three parameters: the number of terms in the recurrence relation, the coefficients in the recurrence relation, and the initial conditions. We use a certain generalization, called a *positive linear recurrence sequence (PLRS)*, which has restrictions on which parameters can be modified. The purpose of the definition of a PLRS is to allow for a generalized version of Zeckendorf's theorem. However, the technical definition of a PLRS and the choices regarding what can be modified may seem arbitrary or

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difficult to understand. This is why we introduce a tiling interpretation of PLRS's that captures the behavior of the generalized Zeckendorf's theorem.

In this paper, the special cases of first order recurrences, the Fibonacci numbers, L-bonacci numbers, second order recurrences, and third order recurrences with positive coefficients are explored. The main result for all PLRS's with positive coefficients is Tiling Interpretation 3. The correctness of this tiling interpretation is shown in Proposition 3. Finally, we conclude with two equivalent ways to extend the main result to allow zero as a coefficient. A preliminary version of these results, presented in a more expository format, appears in [8].

2. Background

2.1. Generalized Zeckendorf's Theorem

We begin by recalling the original Zeckendorf's theorem.

Theorem 1 ([10]). *Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers, when indexing from* $\{1, 2, 3, 5, \dots\}$. *We call this unique decomposition the* Zeckendorf decomposition.

For example, the Zeckendorf decomposition of 12 is $8 + 3 + 1$. In the literature, there exist many generalizations of Zeckendorf's theorem to more sequences than just the Fibonacci numbers. When generalizing this theorem, there are three main types of changes that can be made to the underlying recurrence relation: the depth of the recurrence can be increased, the coefficients can be changed, and the initial conditions can be changed. However, when making a generalization of Zeckendorf's theorem, we must be careful to consider what properties of the original theorem we wish to preserve. For example, in his original paper [10], Zeckendorf also had a similar result for Lucas numbers which established existence of decompositions. However, uniqueness of decompositions was lost–some numbers have two representations. The Lucas numbers will be discussed further in Section 3.8. For additional generalizations of Zeckendorf's theorem see [5, 6, 7].

Our priority is the preservation of unique decompositions. This property is what allows these sequences to be used as bases of enumeration (see [2, 3]) and it will be essential for our development of tiling representations. Ultimately, the generalized Zeckendorf's theorem that we use here is from [9]. This allows us great flexibility with respect to the depth of the recurrence and the coefficients, while the trade off is that we lose control over the initial conditions, which are forced to be specific values depending on the coefficients.

The following definition establishes exactly for which sequences we have a generalized Zeckendorf's theorem.

Definition 1 ([9])**.** We say a sequence $\{h_n\}_{n=1}^{\infty}$ of positive integers is a *Positive Linear Recurrence Sequence (PLRS)* if the following properties hold:

1. *Recurrence relation:* There are non-negative integers $L, c_1, \ldots, c_L$ such that

$$h_{n+1} \;=\; c_1 h_n + \cdots + c_L h_{n+1-L},$$

with L, c_1 and c_L positive.

2. *Initial conditions:* $h_1 = 1$, and for $1 \leq n < L$ we have

$$h_{n+1} \;=\; c_1 h_n + c_2 h_{n-1} + \cdots + c_n h_1 + 1.$$

The coefficient c_1 cannot be zero to prevent the situation where there are independent subsequences interspersed with each other.[2] The coefficient c_L must be positive to prevent the inclusion of an arbitrary number of coefficients of zero at the end, which would cause L to not be uniquely determined. The "+1" forces the PLRS to grow quickly enough so that there are no repeated terms. Now let us see an example of a PLRS.

Example 1. A PLRS is defined by its coefficients. Suppose that there are $L = 3$ coefficients, $c_1 = 1$, $c_2 = 4$, and $c_3 = 9$. Next, we determine the initial conditions. We always have $h_1 = 1$. Then, by the definition, $h_2 = c_1 h_1 + 1 = 2$, and $h_3 = c_1 h_2 + c_2 h_1 + 1 = 7$. Now, we have three initial conditions, which are sufficient to use the recurrence relation $h_{n+1} = c_1 h_n + c_2 h_{n-1} + c_3 h_{n-2}$. Thus, $h_4 = c_1 h_3 + c_2 h_2 + c_3 h_1 = 24$. Repeatedly applying the recurrence relation allows us to generate the PLRS $\{1, 2, 7, 24, 70, 229, \ldots\}$.

A decomposition of a positive integer N is a sum of positive integers that sum to N. The decomposition is a formal object in the sense that which numbers are summed together, and how many times, is essential information. (We then write each unique summand only once, multiplied by the appropriate coefficient.) In order to get unique decompositions, Zeckendorf's theorem gives a decomposition rule, which is that only nonconsecutive Fibonacci numbers can be used. Shortly, we will define a *legal decomposition*, which is a decomposition that obeys certain rules designed to create unique decompositions. The definition generalizes the rule for the Fibonacci numbers, by making use of *decomposition blocks*. A decomposition block is an ordered sequence of i coefficients a_i (for $1 \leq i \leq L$), where the coefficients are multipliers for a subsection of the PLRS. There are c_i distinct decomposition blocks of length i. (Thus the total number of distinct decomposition blocks is $c_1 + c_2 + \cdots + c_L$.)

[2]For example, if we had the recurrence relation $h_{n+1} = 2h_{n-1}$, only every other term is related. Initial conditions h_1 and h_2 generate the sequence $\{h_1, h_2, 2h_1, 2h_2, 4h_1, 4h_2, 8h_1, 8h_2, \ldots\}$, which is really just two independent sequences, $\{h_1, 2h_1, 4h_1, 8h_1, \ldots\}$ and $\{h_2, 2h_2, 4h_2, 8h_2, \ldots\}$ that alternate.

Definition 2 ([9])**.** We call a decomposition $\sum_{i=1}^{m} a_i h_{m+1-i}$ of a positive integer N a *legal decomposition* if $a_1 > 0$, the other $a_i \geq 0$, and one of the following two conditions holds:

1. We have $m < L$ and $a_i = c_i$ for $1 \leq i \leq m$.

2. There exists $s \in \{1, \ldots, L\}$ such that

$$a_1 \;=\; c_1,\; a_2 \;=\; c_2,\; \ldots,\; a_{s-1} \;=\; c_{s-1} \text{ and } a_s < c_s,^{3}$$

   $a_{s+1}, \ldots, a_{s+\ell} = 0$ for some $\ell \geq 0$, and the remaining decomposition $\sum_{i=s+\ell+1}^{m} a_i h_{m+1-i}$ is legal or empty[4].

At the heart of this definition is the second condition, which states that if there are $s \in \{1, \ldots, L\}$ coefficients used in a decomposition block (the a_i's), then the first $s-1$ of those must match the first $s-1$ coefficients used to generate the PLRS (the c_i's), and for the sth coefficient, $a_s < c_s$. Also, the remaining amount left to decompose must decompose legally as well, after a gap of $\ell \geq 0$ terms in the sequence. Only the second condition can be applied repeatedly within the decomposition of one number, since it allows for the remainder of the decomposition to be calculated recursively. If the first condition is used, the decomposition must end. The first condition states that all the m coefficients used in the decomposition block (the a_i's) must match the first m coefficients used to generate the PLRS (the c_i's).

One way to think about a decomposition block is like a mold with vertical chambers that we can fill up, where each position (vertical chamber) has to be filled in order to begin filling the next position (vertical chamber). Then, once one decomposition block is ended, a new one can begin. Let us see an example.

Notation 1. We will use the notation $[c_1, \ldots, c_L]$, which is the collection of all L coefficients, to represent the PLRS $h_{n+1} = c_1 h_n + \cdots + c_L h_{n+1-L}$. We may also use letters $[a, b, c, \ldots]$ to represent coefficients, to avoid using subscripts.

Example 2. As in Example 1, define a PLRS by the coefficients $[c_1, c_2, c_3] = [1, 4, 9]$. The PLRS is $\{1, 2, 7, 24, 70, 229, \ldots\}$. Suppose we want to create a decomposition for 21. We start a decomposition block with the largest number from the sequence that is less than or equal to the number we wish to decompose. In the case of 21, the largest number in the sequence less than or equal to it is 7. The first coefficient is 1, so we can only use up to one 7 within a decomposition block. So we move to the previous term in the sequence, 2. We can use up to four 2's,

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[3]Clarifying this in the case of small s, if $s = 1$, then the condition is $a_1 < c_1$. If $s = 2$, then the condition is $a_1 = c_1$ and $a_2 < c_2$. If $s \geq 3$, then the condition is $a_1 = c_1, a_2 = c_2, \ldots, a_{s-1} = c_{s-1}$ and $a_s < c_s$.

[4]That the remaining decomposition is legal or empty was originally written as $\{b_i\}_{i=1}^{m-s-\ell}$ (with $b_i = a_{s+\ell+i}$) is legal or empty.

since the second coefficient is 4. As $21 - 7 - 4 \cdot 2 = 6 \geq 0$, we use all four 2's in the decomposition. Then, we go to the third position in the decomposition block, where we can use up to nine 1's. To complete the decomposition, we only need to use six of the 1's. So we get that the decomposition of 21 is $7 + 4 \cdot 2 + 6 \cdot 1$. See Figure 1 for a visual interpretation of how the coefficients fit into the decomposition block like a mold.

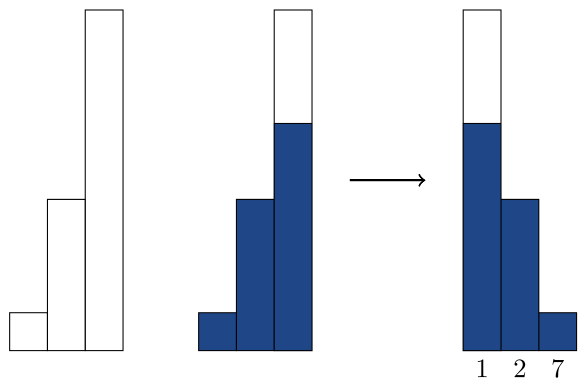


Figure 1: The figure on the left is an empty "mold" that can be used to visualize a decomposition block for a PLRS generated by the coefficients $[1, 4, 9]$. The figure in the middle is representing the particular coefficients $1, 4, 6$ that are used in the decomposition of the number 21. Each coefficient used in a decomposition must be maximized in order for the next coefficient to be used. The figure on the right shows the decomposition blocked flipped, so that the decomposition coefficients (represented by the height of the blue region) are in the correct order for the sequence when written from left to right. Lastly, when this decomposition block's coefficients are applied to the first three sequence terms $1, 2, 7$, we see that the number $6 \cdot 1 + 4 \cdot 2 + 1 \cdot 7 = 21$ is represented.

Multiple decomposition blocks may be necessary. For 134, it has decomposition $[70 + 2 \cdot 24] + [7 + 4 \cdot 2 + 1]$; the first decomposition block has coefficients $1, 2$ which are dominated by $1, 4$ and the second decomposition block has coefficients $1, 4, 1$ which are dominated by $1, 4, 9$. This shows why decomposition blocks need to be able to terminate early, because if we forced the first decomposition block to be $1, 2, 0$ (which is dominated by $1, 4, 9$) then we would not have been able to use any 7's in our decomposition, which we do need.

Now, we state the generalized Zeckendorf's theorem for PLRS's. It is originally due to [4] and this formulation is from [9].

Theorem 2 ([4, 9])**.** *Let $\{h_n\}_{n=1}^{\infty}$ be a Positive Linear Recurrence Sequence. Then there is a unique legal decomposition for each positive integer N.*

2.2. Existing Tiling Interpretations

We recall a well-known tiling interpretation of the Fibonacci numbers (see Chapter 1 in [1]). We can show that f_n counts the number of ways to tile a $1 \times n$ board with 1×1 squares $\boxed{}$ and 1×2 dominoes $\boxed{}$.

First, we check the recurrence relation is satisfied. Note that f_{n+1} counts the number of ways to tile a $1 \times (n+1)$ board with squares and dominoes. (The # symbol indicates the number of ways to tile the undetermined portion of the following board.)

$$f_{n+1} = \# \underbrace{\boxed{\quad\quad \cdots \quad\quad}}_{n+1}$$

Note that f_n counts the number of ways to tile a $1 \times n$ board with squares and dominoes. This is the same as the number of ways to tile a $1 \times (n+1)$ board with squares and dominoes, where the final tile is required to be a square (in red).

$$f_n = \# \underbrace{\boxed{\quad\quad \cdots \quad\quad}}_{n}\boxed{}$$

Note that f_{n-1} counts the number of ways to tile a $1 \times (n-1)$ board with squares and dominoes. This is the same as the number of ways to tile a $1 \times (n+1)$ board with squares and dominoes, where the final tile is required to be a domino (in red).

$$f_{n-1} = \# \underbrace{\boxed{\quad\quad \cdots \quad\quad}}_{n-1}\boxed{}$$

Now, consider that for a $1 \times (n+1)$ board, any valid tiling must have either a square or a domino as the final tile. These two possibilities partition all possible tilings. So, we conclude that this tiling interpretation satisfies the recurrence relation $f_{n+1} = f_n + f_{n-1}$.

Second, we verify that the initial conditions are correct. For a 1×1 board, there is one way to tile it, with a square $\boxed{}$, which gives $f_1 = 1$. For a 1×2 board, there are two ways to tile it, with two squares $\boxed{}\boxed{}$ or with one domino $\boxed{}$, which gives $f_2 = 2$.

Consider the generalization where we introduce positive coefficients $c_i \geq 1$

$$h_{n+1} = c_1 h_n + c_2 h_{n-1} + \cdots + c_L h_{n+1-L}.$$

To represent a coefficient of $c_1 = 2$, we would use 2 colors for the 1×1 squares $\boxed{}$, $\boxed{}$. To represent a coefficient of $c_1 = 3$, we would use 3 colors for the 1×1

squares , , . In general, to represent a coefficient of $c_1 = i$, we would use i colors for the 1×1 squares , , , . . . , .

Moving on to the second coefficient, To represent a coefficient of $c_2 = 2$, we would use 2 colors for the 1×2 dominoes , . To represent a coefficient of $c_2 = 3$, we would use 3 colors for the 1×2 dominoes , , . In general, to represent a coefficient of $c_2 = i$, we would use i colors for the 1×2 dominoes , , , . . . , .

Generalizing this to all coefficients, to represent any positive integer c_i, we can use c_i colors for the $1 \times i$ tiles. To represent a coefficient of zero, $c_i = 0$, there are zero tiles of size $1 \times i$ to use. This appears as Combinatorial Theorem 4 in [1].

Lastly, we can use a variety of initial conditions by implementing special rules for the first tile, called *phases*, as in Combinatorial Theorem 8 in [1]. We will discuss phases in more detail as needed.

3. The Decomposition Tiling Interpretation

In this section, we introduce *decomposition tilings*, which are a modification of the existing tiling interpretations we saw in Section 2.2. By designing decomposition tilings so that they correspond to how decomposition blocks work, they capture the behavior of the generalized Zeckendorf's theorem. We start with first order recurrences in Section 3.1 to introduce decompositions and our decomposition tiling interpretation. We show how to extend this tiling interpretation to the Fibonacci numbers in Section 3.2. We then generalize our tilings to a simple generalization of the Fibonacci numbers, which we call L-bonacci numbers, in Section 3.3. There, we check that our work is correct, in the sense that the tiling interpretation corresponds to the generalized Zeckendorf's theorem, with Proposition 2. L-bonacci numbers are only generated by coefficients of 1. So we extend our tiling interpretation to second order and third order recurrences with arbitrary positive coefficients in Sections 3.4 and 3.5. Then, we generalize our results to all PLRS's with positive coefficients in Section 3.6. We also check that our work is correct again with Proposition 3. Finally, we discuss two ways to modify the tiling interpretation to permit coefficients of zero in Section 3.7.

3.1. First Order Recurrences

All first order PLRS's can be written as $h_{n+1} = c_1 h_n$, and have the initial condition of $h_1 = 1$. Since there is only one coefficient, we can forgo the subscript and write $c = c_1$. Ignoring the case of the trivial sequence generated by $c = 1$, we note that $c = 2$ generates the sequence $\{1, 2, 4, 8, 16, 32, \ldots\}$, that $c = 3$ generates the sequence $\{1, 3, 9, 27, 81, 243, \ldots\}$, that $c = 4$ generates the sequence $\{1, 4, 16, 64, 256, 1024, \ldots\}$, and in general, c generates the sequence $\{1, c, c^2, c^3, c^4, c^5, \ldots\}$.

Consider the following example, where we express 248 in other bases. In base-2 (binary), it is 11111000_2, which has decomposition $1 \cdot 2^7 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$. In base-3 (ternary), it is 100012_3, which has decomposition $1 \cdot 3^5 + 0 \cdot 3^4 + 0 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3^1 + 2 \cdot 3^0$. In base-4 (quaternary), it is 3320_4, which has decomposition $3 \cdot 4^3 + 3 \cdot 4^2 + 2 \cdot 4^1 + 0 \cdot 4^0$.

A basic property of a base-c representation of a number is that for each power of c, it can be multiplied by a coefficient in $\{0, 1, \ldots, c-1\}$. A coefficient cannot be c or higher, because then a larger power of c would be used instead. A straightforward way to represent numbers in base-c using tilings would be to use c squares, each with a *filter*, which acts as a multiplier of $\{\times 0, \times 1, \ldots, \times(c-1)\}$.[5] For the purpose of enumerating the total number of possible tilings on a board of a given length, these filters can correspond to a color, and are counted as described in Section 2.

Example 3 (Binary Tilings)**.** To represent the binary decomposition of a number with tilings, we will be tiling a board that is labelled with the sequence of powers of 2.

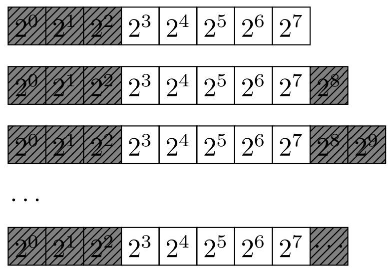
2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	$\cdots$

We saw that $248_{10} = 11111000_2 = 1 \cdot 2^7 + 1 \cdot 2^6 + 1 \cdot 2^5 + 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$. To capture this on our board, we use transparent tiles to represent a coefficient of 1 and opaque tiles to represent a coefficient of 0 as follows.

2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	$\cdots$

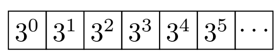
Then the way to "read" this tiling is to ignore anything on the board that is covered by an opaque tile (since the opaque tile is blocking it from being read), and then add all of the uncovered entries. As we know that binary representations of numbers are unique, we just need to make sure that there is a unique tiling representation of any number. Now with this tiling interpretation, any positions that are labelled with 2^n with $n > 7$ must be covered by an opaque tile. As a result, all of the following tilings can be considered equivalent.

[5]These filters are inspired by the double letter and triple letter score spaces on a Scrabble game board (which are blue and green respectively).

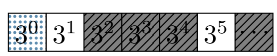


As we want a unique tiling representation, we will use the last board, which is semi-infinite (infinite in one direction), as only this will be long enough for all numbers. However, in practice, we can omit discussion of the length of the board (and the ellipsis), as long as all non-opaque tiles are shown, with the understanding that any drawings are equivalent to a semi-infinite board.

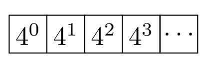
Example 4 (Ternary Tilings)**.** To represent the ternary decomposition of a number with tilings, we will be tiling a board that is labelled with the sequence of powers of 3.



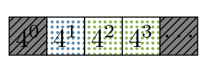
We saw that $248_{10} = 100012_3 = 1 \cdot 3^5 + 0 \cdot 3^4 + 0 \cdot 3^3 + 0 \cdot 3^2 + 1 \cdot 3^1 + 2 \cdot 3^0$. To capture this on our board, we use transparent tiles with a light blue $\times 2$ filter to represent a coefficient of 2, transparent tiles to represent a coefficient of 1 and opaque tiles to represent a coefficient of 0 as follows.



Example 5 (Quaternary Tilings)**.** To represent the ternary decomposition of a number with tilings, we will be tiling a board that is labelled with the sequence of powers of 4.

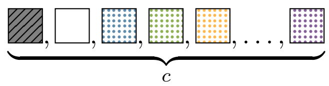


We saw that $248_{10} = 3320_4 = 3 \cdot 4^3 + 3 \cdot 4^2 + 2 \cdot 4^1 + 0 \cdot 4^0$. To capture this on our board, we use transparent tiles with a light green $\times 3$ filter to represent a coefficient of 3, we use transparent tiles with a light blue $\times 2$ filter to represent a coefficient of 2, transparent tiles to represent a coefficient of 1 and opaque tiles to represent a coefficient of 0 as follows.



We can generalize the results from this section in the following result. While simple, it lays an important foundation for generalizing to more complicated and interesting sequences.

Tiling Interpretation 1 (Decomposition Tilings for First Order Sequences)**.** Consider any PLRS generated by $[c]$, i.e., defined by $h_{n+1} = ch_n$, with $c > 1$. Construct a semi-infinite strip and label each space by the terms of the PLRS, beginning with $h_1 = 1$.[6] Tile the strip with 1×1 squares

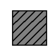
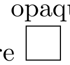
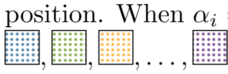


We show that our tiling interpretation corresponds to the unique decompositions given by the generalized Zeckendorf's theorem.

Proposition 1. *The unique decomposition of any positive integer guaranteed by the generalized Zeckendorf's theorem (Theorem 2) into a sum of terms of a PLRS, generated by one coefficient $c > 1$, has a one-to-one correspondence with a decomposition tiling representation given by Tiling Interpretation 1.*

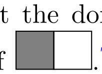
Proof. A first order PLRS is $h_{n+1} = ch_n$. Let N be the number we wish to represent. Begin with a semi-infinite board and label each position as follows:

c^0	c^1	c^2	c^3	c^4	c^5	$\cdots$

Recall that every positive integer N has a unique base-c representation for all natural numbers $c > 1$. So we can write $N = \alpha_0 + \alpha_1 c + \alpha_2 c^2 + \alpha_3 c^3 + \alpha_4 c^4 + \alpha_5 c^5 + \cdots$ with a unique sequence of $\{\alpha_i\}_i$. Then, when $\alpha_i = 0$, use an opaque square  to cover that position. When $\alpha_i = 1$, use a transparent square  to cover that position. When $\alpha_i = 2, 3, 4, \ldots, c-1$, use a transparent square with a colored filter  to cover that position, where each colored filter represents a multiplier of $\times 2, \times 3, \times 4, \ldots, \times(c-1)$. As this tiling is in one-to-one correspondence with the unique base-c representation, it is unique. $\square$

3.2. Fibonacci Numbers

In this section we combine the existing tiling interpretation of the Fibonacci numbers, using squares and dominoes, with Zeckendorf's theorem.

From Section 2.2, standard tilings that correspond to Fibonacci numbers use squares and dominoes in one color. Now, we modify the tiles by applying the concept of transparency. Let the squares be opaque , and let the dominoes be opaque for their first half and transparent for their second half .[7] This

[6] We will later use boards where there is a zeroth position on the board labeled with 0 before the positions $i > 0$ are labelled with h_i. The board used here without a 0 is equivalent to a board with a 0, when we require that an opaque square is used to cover to 0.

[7] Note that the squares have purely cosmetic diagonal black lines while all non-square tiles do not. This is to help distinguish between spaces on boards that are covered by opaque squares and by opaque parts of dominoes or other tiles.

$0 = \sum_{\varnothing}$
$1 = 1$
$2 = 2$
$3 = 3$
$4 = 3 + 1$
$5 = 5$
$6 = 5 + 1$
$7 = 5 + 2$
$8 = 8$
$9 = 8 + 1$
$10 = 8 + 2$
$11 = 8 + 3$
$12 = 8 + 3 + 1$

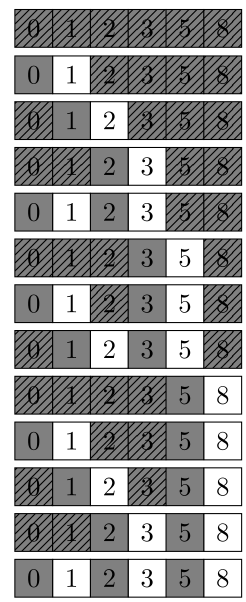


Figure 2: Examples of decomposition tilings for the Fibonacci numbers, i.e., the PLRS generated by $[1, 1]$.

modification captures the behavior of the decomposition rule, which says that no consecutive Fibonacci numbers may be used. Since the dominoes are half-opaque and half-transparent, this prevents any consecutive Fibonacci numbers from ever being used.

We label the board that we are tiling with a zero, followed by the Fibonacci numbers as follows.

0	1	2	3	5	8

For first-order recurrences, we did not use a space with a zero. However, it is necessary now in order to allow a domino's transparent half to be placed over the first sequence term 1, so that it can be used in a decomposition. For examples of decomposition tilings, see Figure 2. Note that $\sum_{\varnothing}$ denotes the empty sum, which is equal to the additive identity, 0.

3.3. *L*-bonacci Numbers

One generalization of the Fibonacci numbers is the L-bonacci numbers, defined by $h_{n+1} = h_n + h_{n-1} + \cdots + h_{n-L+1}$. Note that these are those PLRS's where all

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing}\\
1 &= 1\\
2 &= 2\\
3 &= 2+1\\
4 &= 4\\
5 &= 4+1\\
6 &= 4+2\\
7 &= 7\\
8 &= 7+1\\
9 &= 7+2\\
10 &= 7+2+1\\
11 &= 7+4\\
12 &= 7+4+1
\end{aligned}
$$

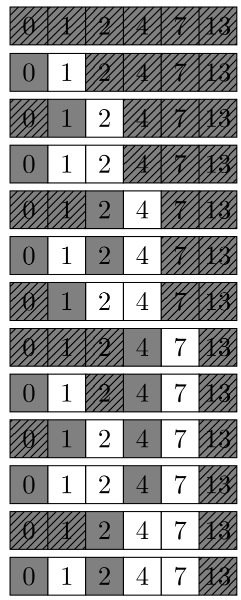
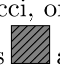
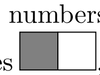
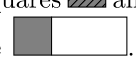


Figure 3: Examples of decomposition tilings for the tribonacci numbers, i.e., the PLRS generated by $[1, 1, 1]$.

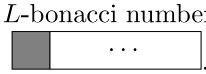
coefficients are 1. Again, we use the initial conditions given by the definition of a PLRS.

First, we consider the 3-bonacci, or tribonacci, numbers defined by $t_{n+1} = t_n + t_{n-1} + t_{n-2}$. Again we use squares  and dominoes , but we add in trominoes with the first third opaque . This allows two (but not three or more) consecutive tribonacci numbers to be used.

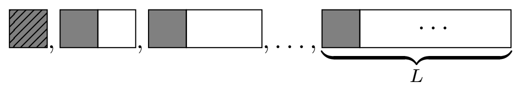
Again we use labels on the board that we are tiling:

0	1	2	4	7	13

For examples of decomposition tilings, see Figure 3.

We can extended this method to all L-bonacci numbers defined by $h_{n+1} = h_n + \cdots + h_{n+1-L}$, using tiles of the form . We state this precisely in the following result, which we also prove aligns with the existing definition of decomposition blocks.

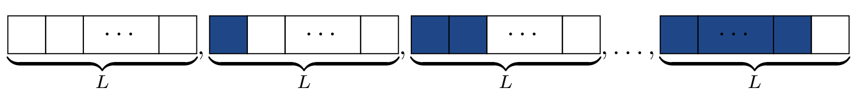
Tiling Interpretation 2 (Decomposition Tilings for L-bonacci Sequences)**.** Consider any PLRS generated by $[\underbrace{1, \ldots, 1}_{L}]$. Construct a semi-infinite strip, and label the initial 1×1 space with a 0. Then label each subsequent space by the terms of the PLRS, beginning with $h_1 = 1$. Tile the strip with $1 \times i$ tiles where the leftmost 1×1 part of each tile is opaque, and the remaining $1 \times (i-1)$ part of each tile is transparent, for all $i \in \{1, \ldots, L\}$, i.e.,



We show that our tiling interpretation corresponds to the unique decompositions given by the generalized Zeckendorf's theorem.

Proposition 2. *The unique decomposition of any positive integer guaranteed by the generalized Zeckendorf's theorem (Theorem 2) into a sum of terms of a PLRS, generated by L coefficients of 1, has a one-to-one correspondence with a decomposition tiling representation given by Tiling Interpretation 2.*

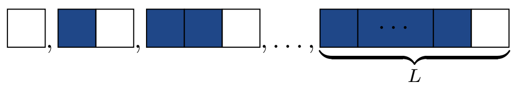
Proof. We draw directly on the definition of a decomposition block, to show a correspondence between decomposition blocks and the tiles that can be used. Decomposition blocks were introduced in the context of legal decompositions in Definition 2. As a decomposition block's coefficients act as multipliers for subsections of the PLRS, they are equivalent to using the tilings that we have defined in Tiling Interpretation 2 as we show here. Note that in decompositions blocks, a blue color means that it is being used, while white is not. The first step is to consider all possible decomposition blocks. We create all possible decomposition blocks by taking an empty mold, where the height of each position corresponds to a coefficient used to generate the PLRS. Since we are considering only L-bonacci numbers, all coefficients are 1, and the decomposition blocks (before any modifications) are



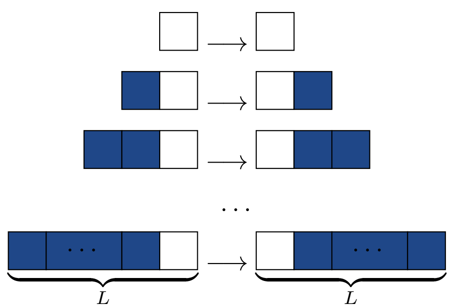
However, we can end a decomposition block early. Let a_i be a decomposition block coefficient and c_i be a PLRS coefficient. We model the decomposition blocks after the second condition of Definition 2, which says that there exists $s \in \{1, \ldots, L\}$ such that $a_1 = c_1,\ a_2 = c_2,\ \ldots,\ a_{s-1} = c_{s-1},\ a_s < c_s$, and $a_{s+1}, \ldots, a_{s+\ell} = 0$ for some $\ell \geq 0$. Thus, if the first $s-1$ coefficients are matched, the decomposition block cannot have a width of just $s-1$, it also must have a coefficient $a_s = 0$, to satisfy $a_s < c_s$.[8] In the case of the L-bonacci numbers, there is no possibility of coefficients being greater than 1, so there will never be a partially full location. Thus

[8]However, if the final portion of a decomposition mold is partially full, there is no need to add additional empty positions following it. This does not apply here since all coefficients are 1.

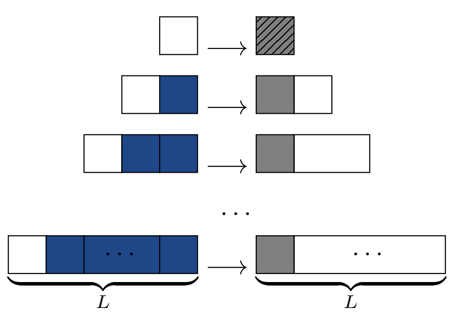
all decomposition blocks will end in an empty space. The minimized decompositions blocks then become



Finally, we reverse all of the decomposition blocks, because according to the definition, the PLRS $\{h_n\}_{n=1}^{\infty}$ and the decomposition blocks (with coefficients a_i) increment in opposite directions (since a decomposition is written $N = \sum_{i=1}^{m} a_i h_{m+1-i}$). By reversing the decomposition blocks, we can orient them correctly on the PLRS going from left to right. The reversal is as follows.



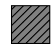
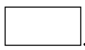
Now that we have the reversed decomposition blocks, we can see that they correspond to the tilings for L-bonacci numbers, by mapping empty (white) squares to opaque (gray) squares, and mapping filled (blue) squares to transparent squares.



Since we are using tilings to represent decomposition blocks over the same PLRS (with the addition of an initial zero), they are interchangeable representations. As the decomposition of a positive integer into decomposition blocks is given uniquely by the generalized Zeckendorf's theorem, the tiling interpretation also faithfully represents the decomposition tiling representation. $\square$

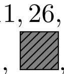
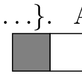
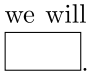
In the generalization to the L-bonacci numbers, we took advantage of the simple decomposition rule. Next, we extend decomposition tilings to include coefficients $c_i > 1$ for second order recurrences.

3.4. Second Order Recurrences

We consider all second order recurrences; they are those generated by coefficients $[a, b]$ for all positive integers a, b. First, we show how to develop $[1, b]$. We already saw $[1, 1]$ which are the Fibonacci numbers. For $[1, 2]$, we get the sequence $\{1, 2, 4, 8, 16, 32, \ldots\}$.[9] We cannot use exactly the same types of tiles as for the Fibonacci numbers, because legal decompositions for this sequence permit an arbitrary number of terms to be used in a row. In addition to the squares  and dominoes  that we used for the Fibonacci numbers, we add transparent dominoes . It is important to understand why we add transparent dominoes and not transparent squares. If we were to include transparent squares, then we would lose uniqueness of the decompositions, as we would be able to create tilings with either the half-opaque and half-transparent domino or an opaque square and a transparent square. On the other hand, if we were to instead add transparent trominoes (or longer tiles), then there would be no way to have just two transparent spaces in a row, with opaque spaces on either side. Using these tiles, examples of the Zeckendorf decomposition tilings are shown in Figure 4.

Observe that there are no transparent dominoes that cover the 0 on the board. Since placing a half opaque-half transparent domino or a transparent domino as the first tile on the board would both contribute a value of 1, we have the rule that any part of a tile that covers a 0 must be opaque. Since 0 only occurs as the first label on the board, this rule only restricts what the initial tile can be. This is where we use phased tiles, where initial conditions are affected by the possibilities for the initial tile only. After the initial tile, the recurrence relation takes over.

The boundaries between tiles in the figures may appear ambiguous at first. However, the greedy algorithm allows us to determine which tiles are used, by proceeding from the rightmost tile which is not an opaque square, and identifying the tile that is the longest, and with the greatest multipliers that are valid, and proceeding recursively.

Next, consider $[1, 3]$, which generates the sequence $\{1, 2, 5, 11, 26, \ldots\}$. Again, we will use the three tiles from the previous case (when $b = 2$), , , and . However, the decomposition rule also allows for a number to be used twice sometimes, for example, the decomposition of 4 is $2 + 1 + 1$. Specifically, from the second condition of Definition 2, within a decomposition block, a coefficient in a subsequent position beyond the first can only be used if all previous positions' coefficients are maximal. So within one decomposition block (and one tile), we

[9] We saw this sequence before; it is also used in the binary decomposition of numbers in Section 3.1. Compare the Zeckendorf tilings of the two, and while they appear very similar at first, the sequence generated by [2] uses only opaque and transparent squares, while the sequence generated by $[1, 2]$ uses opaque squares and two types of dominoes.

$0 = \sum_{\varnothing}$
$1 = 1$
$2 = 2$
$3 = 2 + 1$
$4 = 4$
$5 = 4 + 1$
$6 = 4 + 2$
$7 = 4 + 2 + 1$
$8 = 8$
$9 = 8 + 1$
$10 = 8 + 2$
$11 = 8 + 2 + 1$
$12 = 8 + 4$

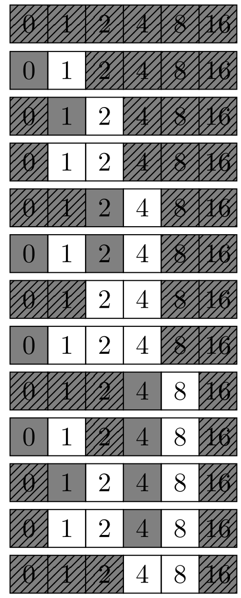
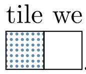
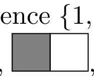
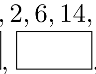
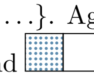
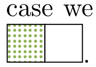
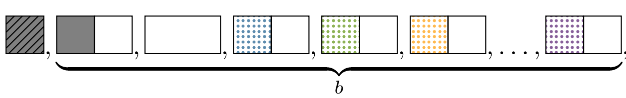


Figure 4: Examples of decomposition tilings for the PLRS generated by $[1, 2]$.

only want a light blue $\times 2$ filter to be used when a coefficient of one (the maximal first coefficient from $[1, 3]$) was already used on the first part of the tile. Since we approach tilings from the largest values first and work recursively, that means that the additional tile we want is a transparent domino, with the left half with a light blue $\times 2$ filter, . The Zeckendorf decomposition tilings are in Figure 5.

Next, consider $[1, 4]$, which generates the sequence $\{1, 2, 6, 14, 38, \ldots\}$. Again, we will use the four tiles from the previous case, , , , and . However, the decomposition rule also allows for a number to be used thrice sometimes. Analogously to the previous case we add a transparent domino, with the left half with a light green $\times 3$ filter, . The Zeckendorf decomposition tilings are in Figure 6.

Generalizing these examples, the decomposition tiling interpretation for any PLRS generated by $[1, b]$ uses tiles



where there is one square, and there are b dominoes, which have filters (if $b > 2$), beginning at $\times 2$ up to $\times(b - 1)$.

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing} \\
1 &= 1 \\
2 &= 2 \\
3 &= 2+1 \\
4 &= 2+1+1 \\
5 &= 5 \\
6 &= 5+1 \\
7 &= 5+2 \\
8 &= 5+2+1 \\
9 &= 5+2+2 \\
10 &= 5+2+2+1 \\
11 &= 11 \\
12 &= 11+1
\end{aligned}
$$

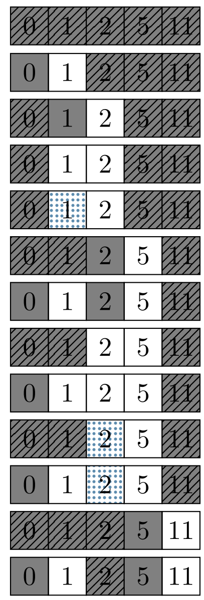
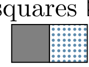
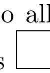
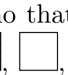
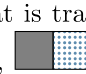
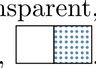


Figure 5: Examples of decomposition tilings for the PLRS generated by $[1, 3]$.

Next, we consider the case of PLRS's generated by $[2, b]$. Naturally, we wish to see if we can use the tiles that were used for $[1, b]$, with any necessary modifications. The key part of the definition of a legal decomposition that is relevant here is that in order to have multiple nonzero coefficients in a row that are part of the same decomposition block (and tile), all coefficients except for the last must be maximized. Since we only are dealing with two coefficients here, then for any domino that has no opaque regions, the rightmost part must have a light blue $\times 2$ filter (from the 2 in $[2, b]$).

First consider $[2, 1]$, which generates the sequence $\{1, 3, 7, 17, 41, \ldots\}$. We use the squares  and dominoes, where we change the rightmost part to have a $\times 2$ filter . Now, we need to add a transparent tile with no filter to allow for decompositions that just use sequence terms once. We use the squares , since they can appear adjacent to each other any number of times, by repeatedly starting new decomposition blocks. Examples of Zeckendorf decomposition tilings using the sequence generated by $[2, 1]$ are in Figure 7.

Next, consider $[2, 2]$. We just need to add another domino that is transparent, with a light blue $\times 2$ filter on the right half. So, we use , , , .

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing} \\
1 &= 1 \\
2 &= 2 \\
3 &= 2+1 \\
4 &= 2+1+1 \\
5 &= 2+1+1+1 \\
6 &= 6 \\
7 &= 6+1 \\
8 &= 6+2 \\
9 &= 6+2+1 \\
10 &= 6+2+2 \\
11 &= 6+2+2+1 \\
12 &= 6+2+2+2
\end{aligned}
$$

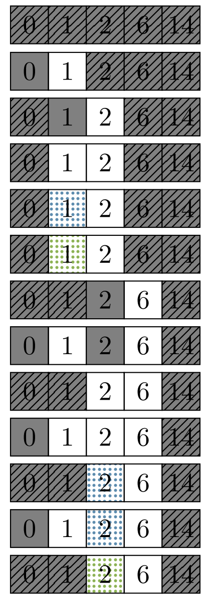
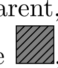
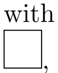
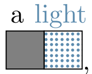
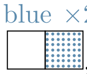
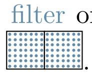
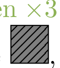
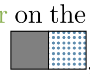
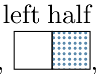
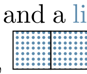
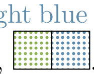


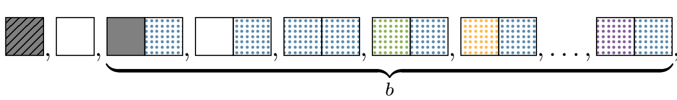
Figure 6: Examples of decomposition tilings for the PLRS generated by $[1, 4]$.

Examples of Zeckendorf decomposition tilings using the sequence generated by $[2, 2]$ are in Figure 8.

Next, consider $[2, 3]$. We just need to add another domino that is transparent, with a light blue $\times 2$ filter on both the left and the right halves. So, we use , , , , . Examples of Zeckendorf decomposition tilings using the sequence generated by $[2, 3]$ are in Figure 9.

Next, consider $[2, 4]$. We just need to add another domino that is transparent, with a light green $\times 3$ filter on the left half and a light blue $\times 2$ filter on the right half. So, the tiles are , , , , , . Examples of Zeckendorf decomposition tilings using the sequence generated by $[2, 4]$ are in Figure 10.

Generalizing this pattern, the decomposition tiling interpretation for any PLRS generated by $[2, b]$ uses tiles



where there are two squares, and there are b dominoes, which have filters (if $b > 2$), beginning at $\times 2$ up to $\times(b - 1)$.

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing}\\
1 &= 1\\
2 &= 1+1\\
3 &= 3\\
4 &= 3+1\\
5 &= 3+1+1\\
6 &= 3+3\\
7 &= 7\\
8 &= 7+1\\
9 &= 7+1+1\\
10 &= 7+3\\
11 &= 7+3+1\\
12 &= 7+3+1+1
\end{aligned}
$$

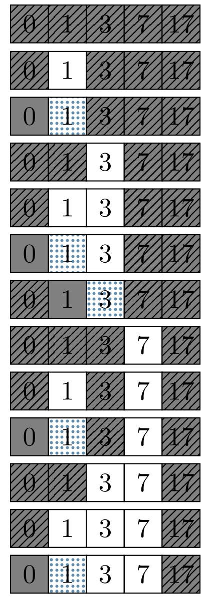


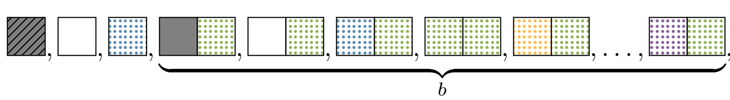
Figure 7: Examples of decomposition tilings for the PLRS generated by $[2, 1]$.

Next, we consider the case of PLRS's generated by $[3, b]$. By the decomposition rule, for a domino, the right half must have a light green $\times 3$ filter. So we can reuse the tiles from the $[2, b]$ case, where we modify the right half of all dominoes to have a light green $\times 3$ filter. We also need to add in a square with a light blue $\times 2$ filter, since we can use that in an unrestricted manner now.

First, consider $[3, 1]$. We use tiles , , , . Examples of Zeckendorf decomposition tilings using the sequence generated by $[3, 1]$ are in Figure 11.

For $[3, 2]$, we use tiles , , , , . For $[3, 3]$, we use tiles , , , , , . For $[3, 4]$, we use tiles , , , , , , .

Analogously to the $[2, b]$ case, the decomposition tiling interpretation for any PLRS generated by $[3, b]$ uses tiles



where there are three squares and there are b dominoes, which (if $b > 2$) have filters on their left half, from $\times 2$ up to $\times(b - 1)$.

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing} \\
1 &= 1 \\
2 &= 1+1 \\
3 &= 3 \\
4 &= 3+1 \\
5 &= 3+1+1 \\
6 &= 3+3 \\
7 &= 3+3+1 \\
8 &= 8 \\
9 &= 8+1 \\
10 &= 8+1+1 \\
11 &= 8+3 \\
12 &= 8+3+1
\end{aligned}
$$

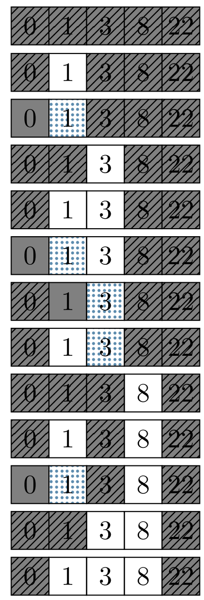
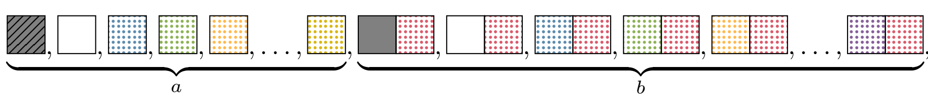


Figure 8: Examples of decomposition tilings for the PLRS generated by $[2, 2]$.

When comparing cases $[1, b]$, $[2, b]$, and $[3, b]$, we see that as the first coefficient increases by one, we make two changes to the set of tiles we use. First, we add an additional transparent square tile with a $\times(a-1)$ filter. Second, we increase the multiplier of the filter on the right half of all dominoes by 1. So, in general, we use the $a + b$ types of tiles



where there are a square tiles available, with multipliers from 0 (opaque) up to $\times(a-1)$. Additionally, there are b domino tiles available, which all have a $\times a$ multiplier on the right half, and multipliers ranging from 0 (opaque) up to $\times(b-1)$.

3.5. Third Order Recurrences with Positive Coefficients

We extend the second order decomposition tiling interpretation to higher order recurrences. Recall that by the definition of a PLRS, the coefficients it is generated by neither start nor end in a zero. So when dealing with first and second order recurrences, there never can be a coefficient of zero. However, when the recurrence is of third order or higher, zero may appear as any of the middle coefficients. Such

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing} \\
1 &= 1 \\
2 &= 1+1 \\
3 &= 3 \\
4 &= 3+1 \\
5 &= 3+1+1 \\
6 &= 3+3 \\
7 &= 3+3+1 \\
8 &= 3+3+1+1 \\
9 &= 9 \\
10 &= 9+1 \\
11 &= 9+1+1 \\
12 &= 9+3
\end{aligned}
$$

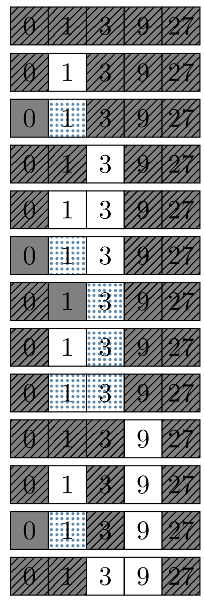
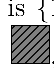
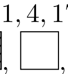
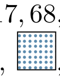
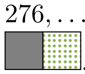
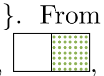
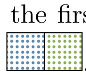
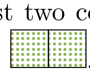


Figure 9: Examples of decomposition tilings for the PLRS generated by $[2, 3]$.

a zero complicates the tiling interpretation slightly, so we first address recurrences with only positive coefficients.

In the case of a second order recurrence $[a, b]$, we used a square tiles and b dominoes. So, we expect the third order case of $[a, b, c]$ to use a square tiles, b dominoes, and c trominoes. Regarding what transparencies and filters to use, we want to reuse the same a squares and b dominoes as in the case of $[a, b]$. Recall that in the case of $[a, b]$, the squares have multipliers from 0 to $a-1$, and the dominoes have a right-half multiplier of a and a left-half multiplier of 0 to $b-1$. So for trominoes, we would like them to have a right-third multiplier of a, a middle-third multiplier of b, and a left-third multiplier of 0 to $c-1$. This is because the definition of a legal decomposition block requires that for a subsequent coefficient to be used within a decomposition block, all previous coefficients have to be maximized.

For example, consider the sequence generated by $[3, 4, 1]$, which is $\{1, 4, 17, 68, 276, \ldots\}$. From the first two coefficients, we use the tiles for $[3, 4]$, , , , , , , . Then, according to the idea we just proposed, we also add in one length 3 tile, specifically, the last two positions will be maximized with the first two coefficients, and the first position will range from a multiplier of 0 up to one less than c. In this case, $c-1$ and 0 are the same, so our one additional

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing}\\
1 &= 1\\
2 &= 1+1\\
3 &= 3\\
4 &= 3+1\\
5 &= 3+1+1\\
6 &= 3+3\\
7 &= 3+3+1\\
8 &= 3+3+1+1\\
9 &= 3+3+1+1+1\\
10 &= 10\\
11 &= 10+1\\
12 &= 10+1+1
\end{aligned}
$$

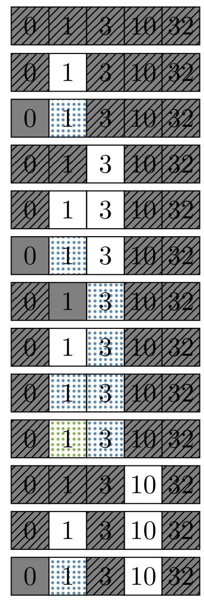
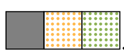


Figure 10: Examples of decomposition tilings for the PLRS generated by $[2, 4]$.

tile will be . As before, we use opaque tiles to represent a coefficient of 0, transparent tiles to represent a coefficient of 1, transparent tiles with a light blue $\times 2$ filter to represent a coefficient of 2, transparent tiles with a light green $\times 3$ filter to represent a coefficient of 3, and transparent tiles with an orange $\times 4$ filter to represent a coefficient of 4. See examples of these tilings in Figure 12.

3.6. The General Decomposition Tiling Interpretation for Positive Coefficients

We can generalize the decomposition tiling representation that we have explored so far to recurrences of any order.

Tiling Interpretation 3 (Decomposition Tilings for Positive Sequences)**.** Consider any PLRS generated by positive coefficients $[a, b, c, \ldots, z] = [c_1, c_2, c_3, \ldots, c_L]$. Construct a semi-infinite strip, and label the first 1×1 space with a 0. Then label each subsequent space by the terms of the PLRS, beginning with 1. To tile the strip, we use the $c_1 + c_2 + c_3 + \cdots + c_L$ tiles shown in Figure 13. For readability, the tiles with filters are labeled above by their multipliers. The only restriction on tile placement is that a tile cannot be placed such that a transparent portion (including with a filter) covers the initial 0.

$$
\begin{aligned}
0 &= \textstyle\sum_{\varnothing} \\
1 &= 1 \\
2 &= 1+1 \\
3 &= 1+1+1 \\
4 &= 4 \\
5 &= 4+1 \\
6 &= 4+1+1 \\
7 &= 4+1+1+1 \\
8 &= 4+4 \\
9 &= 4+4+1 \\
10 &= 4+4+1+1 \\
11 &= 4+4+1+1+1 \\
12 &= 4+4+4
\end{aligned}
$$

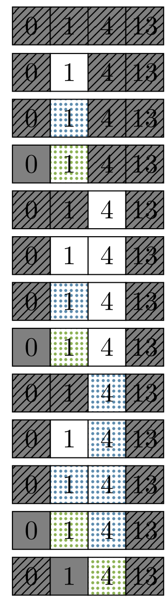


Figure 11: Examples of decomposition tilings for the PLRS generated by $[3, 1]$.

Notice that there are the same number of tiles of a particular length c_i, as the ith coefficient used to generate the PLRS. Specifically, those c_i tiles will all have the same rightmost $i-1$ components, which are, from the right, $c_1, c_2, \ldots, c_{i-1}$, and then the leftmost position ranges from a multiplier of 0 up to $c_i - 1$.

Now we state the following proposition, which establishes the correctness of the connection between this tiling interpretation and the generalized Zeckendorf's theorem.

Proposition 3. *The unique decomposition of any positive integer guaranteed by the generalized Zeckendorf's theorem (Theorem 2) into a sum of terms of a PLRS, generated by positive coefficients, has a one-to-one correspondence with a decomposition tiling representation given by Tiling Interpretation 3.*

Proof. This proof is a generalized version of the proof of Proposition 2. It draws directly on the definition of a decomposition block, to show a correspondence between decomposition blocks and the tiles that can be used. Decomposition blocks were introduced in the context of legal decompositions in Definition 2. As a decomposition block's coefficients act as multipliers for subsections of the PLRS, they are equivalent to using the tilings that we have defined in Tiling Interpretation 3 as we show here. Recall that in decompositions blocks, a blue color means that it is

$1 = 1$

$2 = 1 + 1$

$3 = 1 + 1 + 1$

$4 = 4$

$5 = 4 + 1$

$6 = 4 + 1 + 1$

$7 = 4 + 1 + 1 + 1$

$8 = 4 + 4$

$9 = 4 + 4 + 1$

$10 = 4 + 4 + 1 + 1$

$11 = 4 + 4 + 1 + 1 + 1$

$12 = 4 + 4 + 4$

$13 = 4 + 4 + 4 + 1$

$14 = 4 + 4 + 4 + 1 + 1$

$15 = 4 + 4 + 4 + 1 + 1 + 1$

$16 = 4 + 4 + 4 + 1 + 1 + 1 + 1$

$17 = 17$

$18 = 17 + 1$

$19 = 17 + 1 + 1$

$20 = 17 + 1 + 1 + 1$

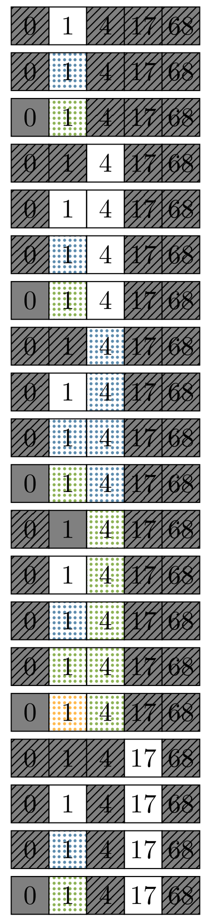


Figure 12: Examples of decomposition tilings for the PLRS generated by $[3, 4, 1]$.

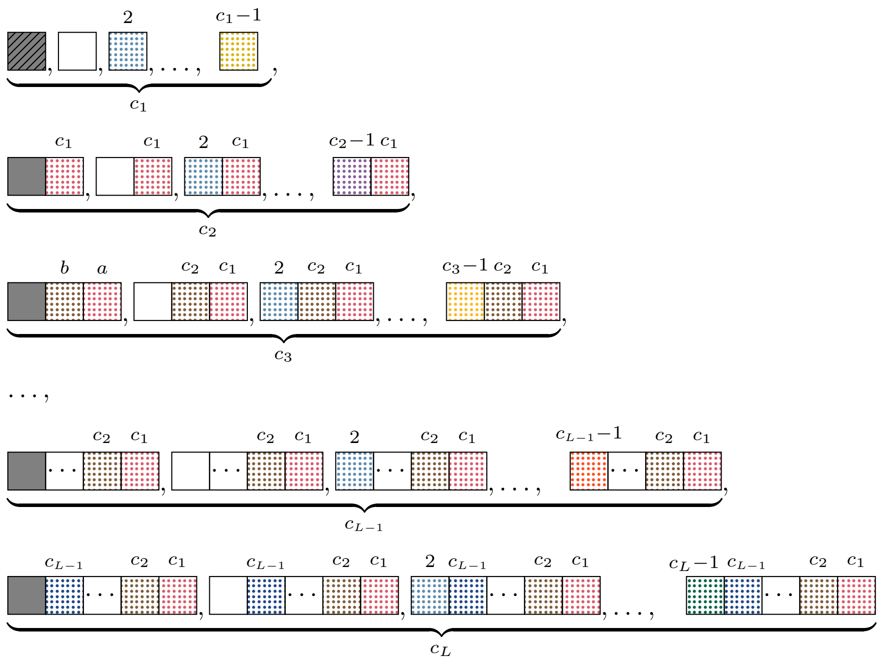


Figure 13: The tiles needed for creating decomposition tilings, for a general PLRS $[c_1, c_2, c_3, \ldots, c_L]$, where $c_i \geq 1$ for all $i \in \{1, \ldots, L\}$.

being used, while white is not. The first step is to consider all possible decomposition blocks. We create all possible decomposition blocks by taking an empty mold, where the height of each position corresponds to a coefficient used to generate the PLRS. The height of the ith column is c_i. The decomposition blocks (before any modifications) are in Figure 14.

However, we can end a decomposition block early. Let a_i be a decomposition block coefficient (that is how high the blue is in a column) and c_i be a PLRS coefficient. We model the decomposition blocks after the second condition of Definition 2, which says that there exists $s \in \{1, \ldots, L\}$ such that $a_1 = c_1,\ a_2 = c_2,\ \ldots,\ a_{s-1} = c_{s-1},\ a_s < c_s$, and $a_{s+1}, \ldots, a_{s+\ell} = 0$ for some $\ell \geq 0$. Thus, if the first $s-1$ coefficients are matched, the decomposition block cannot have a width of just $s-1$, it also must have a coefficient $a_s = 0$, to satisfy $a_s < c_s$. In this general case, whenever there is a coefficient greater than 1, there is the possibility of a partially full location. Thus all decomposition blocks will end in partially full column, or in an empty space. The minimized decompositions blocks are shown in Figure 15.

Finally, we reverse all of the decomposition blocks, because according to the defi-

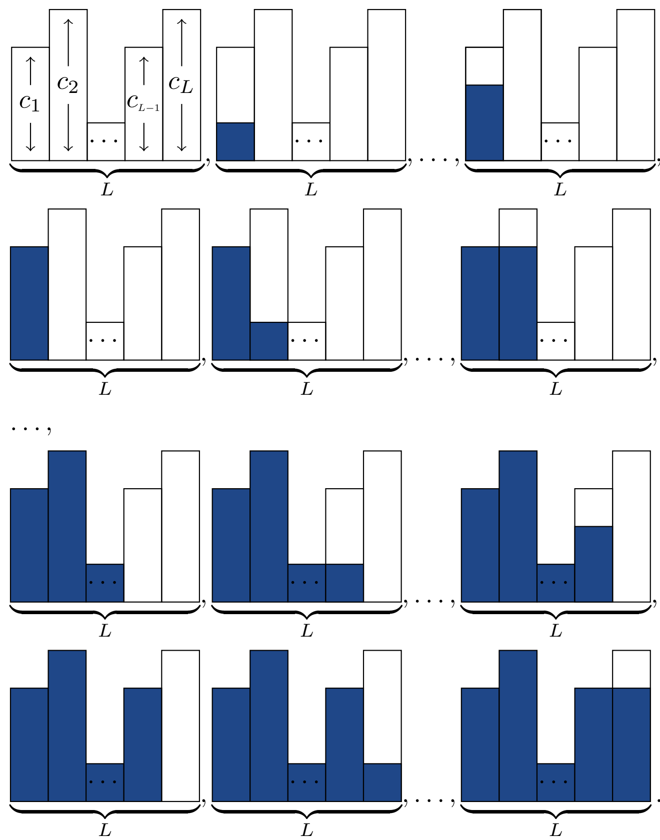


Figure 14: Arbitrary decomposition blocks before any modifications. Note that the height of each column i is c_i. The width of each block is L.

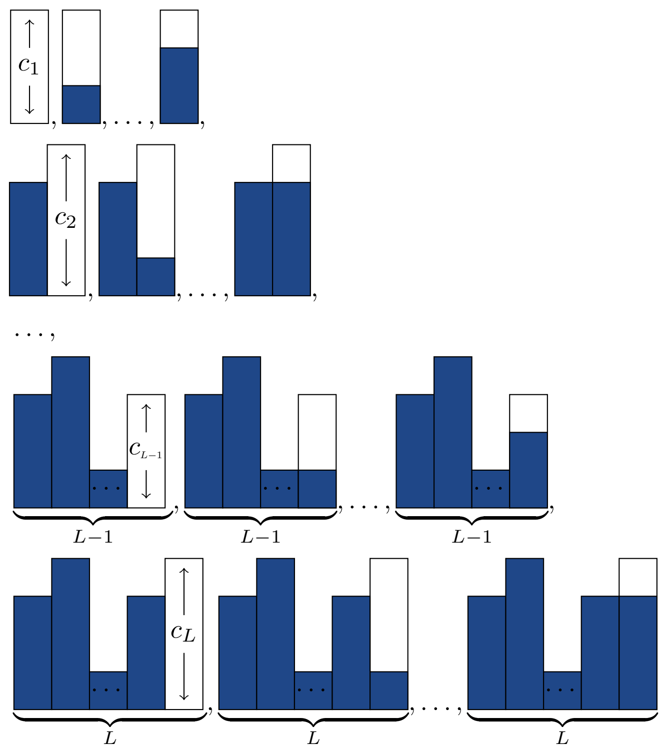


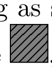
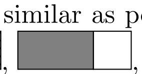
Figure 15: Arbitrary decomposition blocks that have had unnecessary entirely white columns removed. An entirely white column only remains if the column preceding it is completely full (blue).

nition, the PLRS $\{h_n\}_{n=1}^{\infty}$ and the decomposition blocks (with coefficients a_i) increment in opposite directions (since a decomposition is written $N = \sum_{i=1}^{m} a_i h_{m+1-i}$). By reversing the decomposition blocks, we can orient them correctly on the PLRS going from left to right. The reversed decomposition blocks are shown in Figure 16.

Now that we have the reversed decomposition blocks, we can see that they correspond to the tilings for the PLRS generated by $[c_1, \ldots, c_L]$, by mapping white columns to opaque (gray) squares, and mapping blue columns (including partially blue columns) to transparent squares, where the height of each blue column corresponds to the multiplier of the filter. That is, if the blue column is 1 unit high, it becomes a transparent square with no filter, if it is 2 units high, it becomes a transparent square with a light blue $\times 2$ filter, if it is 3 units high, it becomes a transparent square with a light green $\times 3$ filter, etc. The aforementioned decomposition blocks then correspond to the tiles in Figure 17. These tiles are the same as the tiles in Figure 13, which are used in Tiling Interpretation 3.

Since we are using tilings to represent decomposition blocks over the same PLRS (with the addition of an initial zero), they are interchangeable representations. As the decomposition of a positive integer into decomposition blocks is given uniquely by the generalized Zeckendorf's theorem, the tiling interpretation also faithfully represents the decomposition tiling representation. $\square$

3.7. Zero as a Coefficient

Now, let us consider a simple example with a zero, the PLRS generated by $[1, 0, 1]$, which has terms $\{1, 2, 3, 4, 6, 9, 13, 19, 28, 41, \ldots\}$. We want to keep our existing method of tiling as similar as possible. If we naively applied Tiling Interpretation 3, we would use , , since the coefficient $b = 0$, we would get no tiles of size 1×2, and the board we would be tiling is $\boxed{0}\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{6}\boxed{9}\boxed{\cdots}$. However, there would be no tiling to represent the number 1, for example, because the only tile that is non-opaque at any point is the tromino, which is of length 3, so the lowest space it can cover is the space labeled with a 2. We now show two equivalent workarounds: adding additional initial zeros to the beginning of the board, or using phased tilings.

3.7.1. Additional Initial Zeros

With this tiling interpretation, we increase the number of initial zeros on the board from 1 to $1+m$, where m is the maximum number of consecutive zeros in the coefficients used to generate the PLRS. Recall that we have always had a restriction that no tile can be placed with a transparent portion (equivalently a nonzero multiplier) over the initial 0. Here, we extend that restriction to be that no tile can be placed with a transparent portion (equivalently a nonzero multiplier) over any 0 on the

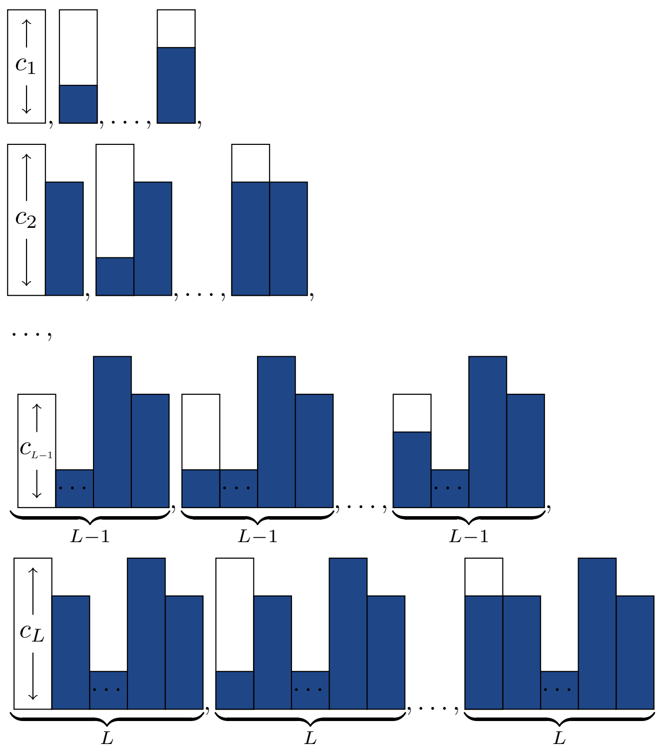


Figure 16: Arbitrary decomposition blocks that have had unnecessary entirely white columns removed, and have been reversed.

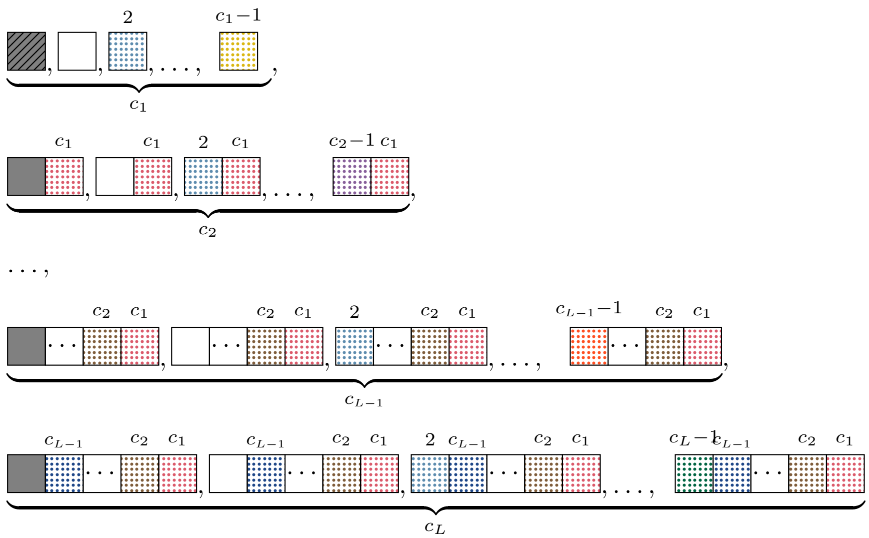
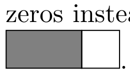
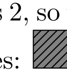
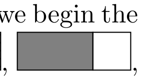
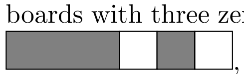
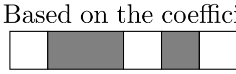


Figure 17: Tiles that correspond to arbitrary decomposition blocks. The numbers above the tiles are the filter multipliers.

board. This is necessary to retain uniqueness of the tilings.

Again, consider the PLRS generated by $[1, 0, 1]$, which is $\{1, 2, 3, 4, 6, 9, 13, 19, \ldots\}$. The maximum number of consecutive zeros is 1, so we begin the boards with two zeros instead of the usual one. Based on the coefficients, we use the tiles:  and . See Figure 18 for examples.

For a second example, consider the PLRS generated by $[1, 0, 1, 0, 0, 2]$, which is $\{1, 2, 3, 5, 8, 12, 19, \ldots\}$. The maximum number of consecutive zeros is 2, so we begin the boards with three zeros. Based on the coefficients, we use the tiles: , , , and . See Figure 19 for examples.

3.7.2. Phased Tilings

One benefit of additional zeros is that all tiles can fit entirely on the board. However, if there is a large number of consecutive zeros, this can occupy a lot of space at the beginning of each tiling without providing much information on any decompositions, since there must always be an opaque tile covering a 0. So, the *phased tilings* method allows us to keep the tilings more compact, by not using any additional zeros. In this case, a phased tiling is exactly the same as its equivalent *additional initial zeros* tiling, with all but the final zero removed. This means that an initial tile may be

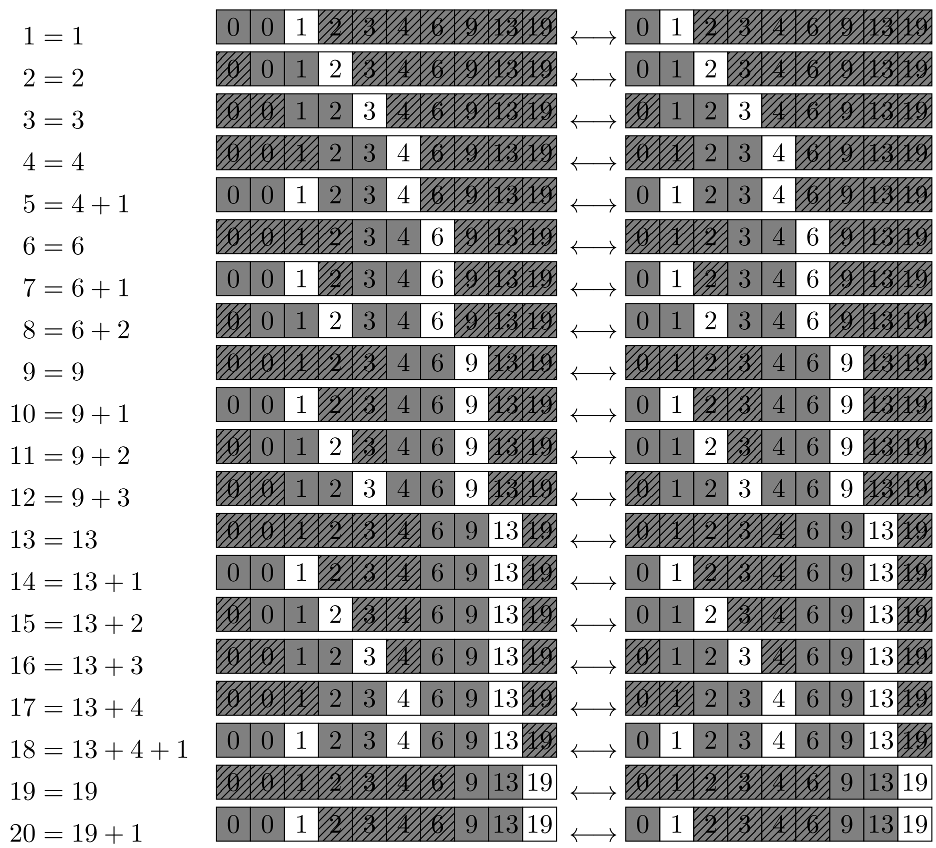


Figure 18: Examples of decomposition tilings for the PLRS generated by $[1, 0, 1]$, using the "additional zeros" interpretation on the left and the "phased tiling" interpretation on the right.

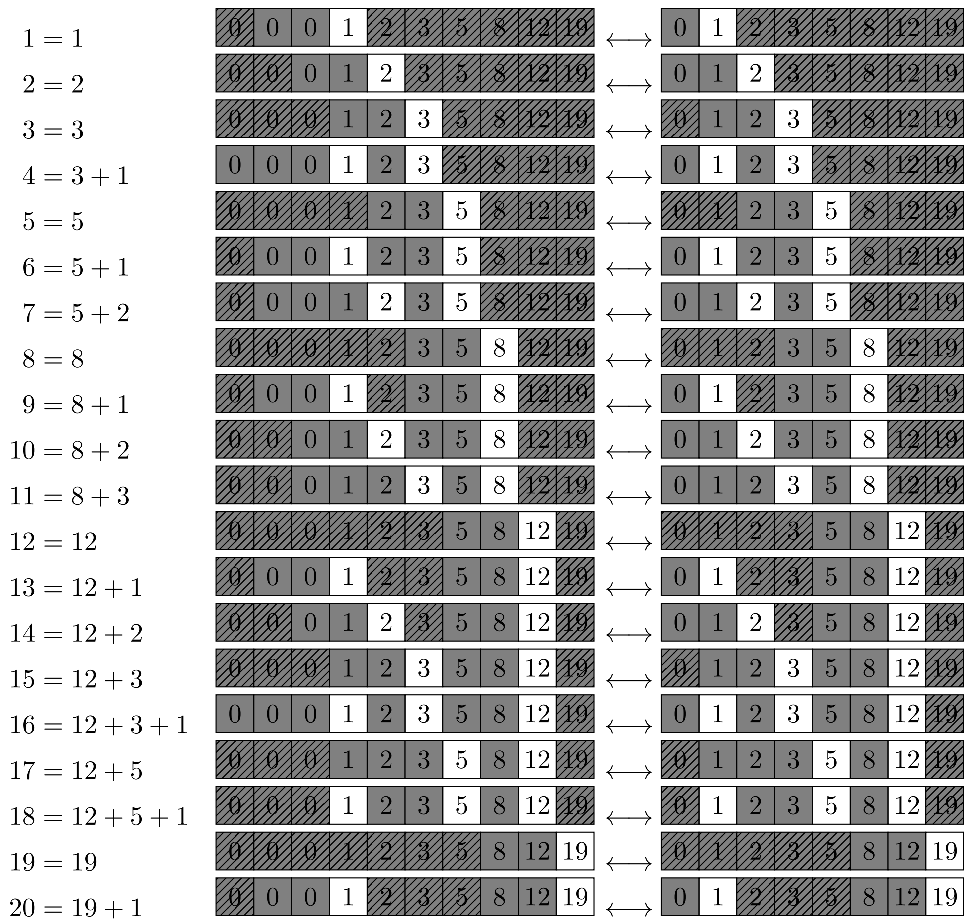
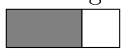
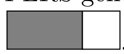
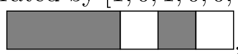
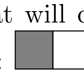
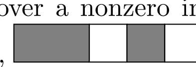
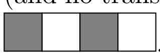


Figure 19: Examples of decomposition tilings for the PLRS generated by $[1, 0, 1, 0, 0, 2]$, using the "additional zeros" interpretation on the left and the "phased tiling" interpretation on the right.

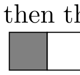
cut off in part at the beginning. The word "phased" just refers to the fact that there are different rules for the first tile (in this case, that it can be a shortened version of another tile). From [1], phases affect the initial conditions of a recurrence relation, but do not cause any further affects once the recurrence relation takes over. Now, we will see the same examples as before, and note how just all but one of the initial zeros are removed from the beginning of the board.

Again, consider the PLRS generated by $[1, 0, 1]$. Based on the coefficients, we use the tiles:  and  in any position. As the tile  could be placed such that the transparent portion is over the 1 on the board (but it would not fit), we cut off the leftmost part, and create a phased tile  that can only be used in the first position. See Figure 18 for examples.

Again, consider the PLRS generated by $[1, 0, 1, 0, 0, 2]$. Based on the coefficients, we use the tiles: , , , and  in any position. We can also use the following phased tiles in the first position only, that we calculate by just removing one unit from the leftmost portion at a time, as long as there remains a transparent portion that will cover a nonzero integer (and no transparent portion will cover a zero either): , , and .[10] See Figure 19 for examples.

3.8. What Goes Wrong with the Lucas Numbers

As discussed in Section 2.1, in order for our tiling interpretation to be unique, we accept the restrictive initial conditions given by the definition of a PLRS. For each recurrence relation, there is only one set of initial conditions we use. If we consider the PLRS generated by $[1, 1]$, we saw in Section 3.2 that the initial conditions must be those for the combinatorial Fibonacci numbers. However, these initial conditions are not the only interesting ones. Consider the Lucas numbers with shifted indices, denoted by $\ell_n = \ell_{n-1} + \ell_{n-2}$ with initial conditions $\ell_1 = 2$ and $\ell_2 = 1$. This gives the sequence $\{2, 1, 3, 4, 7, 11, 18, \ldots\}$. As the Lucas numbers have the same recurrence relation as the Fibonacci numbers, they satisfy the recurrence relation but not the initial conditions in the definition of a PLRS.

We can tile a board that is labelled with 0 and then the sequence of Lucas numbers $\boxed{0 \mid \ell_1 \mid \ell_2 \mid \ell_3 \mid \ell_4 \mid \cdots}$ with the tiles  and , like with the Fibonacci numbers, to represent decompositions of positive integers, due to the following theorem of Zeckendorf.

[10] Note that a phased tile for the first position may arise during the cropping process from multiple of the original tiles.

$0 = \sum_{\varnothing}$
$1 = \ell_2$
$2 = \ell_1$
$3 = \ell_3$
$4 = \ell_4$
$5 = \ell_4 + \ell_2$
$\quad = \ell_3 + \ell_1$
$6 = \ell_4 + \ell_1$
$7 = \ell_5$
$8 = \ell_5 + \ell_2$
$9 = \ell_5 + \ell_1$
$10 = \ell_5 + \ell_3$
$11 = \ell_6$
$12 = \ell_6 + \ell_2$
$\quad = \ell_5 + \ell_3 + \ell_1$

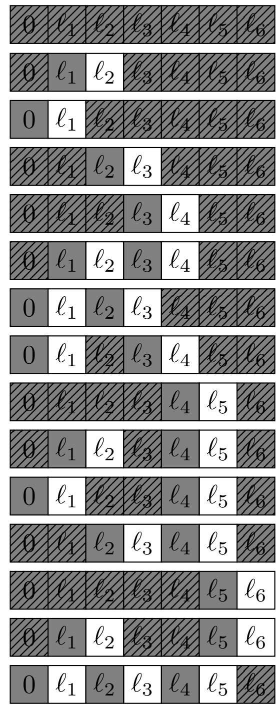


Figure 20: Examples of non-unique decomposition tilings for the Lucas numbers.

Theorem 3 ([10])**.** *Every natural number can be represented by a sum of distinct, nonconsecutive Lucas numbers. The representation is unique, except for the numbers* $\ell_{2v} + 1$ *for* $v = 2, 3, \ldots$.

Now, let us look at examples for the integers 0 through 12, shown in Figure 20. The first two numbers that have non-unique decompositions are 5 and 12 (which occur when $v = 2, 3$ in Theorem 3), which are written in red.

The Lucas numbers have a well-known combinatorial interpretation, as the number of ways to tile a circular n-board with dominoes, which can be in-phase or out-of-phase at a certain location, and squares (see Chapter 2 of [1]). It was not clear to us how to modify this tiling interpretation to allow for decomposition tilings of Lucas numbers. Perhaps instances of non-uniqueness relate to out-of-phase dominoes, since that behavior has no analogue in tilings of n-boards. So we leave the reader with the following question.

Question 1. Can decomposition tilings of PLRS's be modified to allow for other initial conditions, such as those for the Lucas numbers?

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