A TILING INTERPRETATION OF THE q-BINOMIAL COEFFICIENTS

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ABSTRACT. We provide a combinatorial interpretation of the q-binomial and q-multinomial coefficients as counting weighted collections of tiled boards. Using this interpretation, we prove a new q-analogue to Lucas' Theorem and new q-analogues to identities on the sums of integer squares and cubes. Further proofs of known q-identities illustrate the use of proof elements including generating functions, recurrence relations, and sign-reversing involutions, all in the q-binomial context.

1. INTRODUCTION

The q-binomial coefficients are a polynomial generalization of the binomial coefficients. Also referred to as Gaussian binomial coefficients, they arise naturally in many branches of mathematics, including algebra, number theory, statistics, and combinatorics. [4]. (For example, in any finite field F with q elements, $\binom{n}{k}_q$ counts the number of k-dimensional subspaces of an n-dimensional vector space over F.) For any natural number n, we define its q-analog, n_q, as follows: $0_q := 1$ and for $n > 0$, $n_q := 1 + q + q^2 + \cdots + q^{n-1}$. This allows us to define q-factorials: $0_q! := 1$ and $n_q! := 1_q 2_q 3_q \cdots n_q$. Finally, the natural definition of the q-binomial coefficient is

$$\binom{n}{k}_q := \begin{cases} \frac{n_q!}{k_q!(n-k)_q!} & : 0 \le k \le n, \\ 0 & : \text{otherwise.} \end{cases} \tag{1.1}$$

One standard combinatorial interpretation of $\binom{n}{k}$ is that it counts the number of distinct integer partitions that will fit into a box of size $k \times (n-k)$. The corresponding interpretation for $\binom{n}{k}_q$ is partition-counting with the added wrinkle that instead of simply counting up the number of all such partitions, we separate them out by the integer being partitioned. The result is a polynomial in which the coefficient on the q^i term is the number of partitions of the integer i. For example, Figure 1 shows the six partitions that fit into a box of size 2×2: $\{\}, \{1\}, \{2\}, \{1,1\}, \{2,1\}$, and $\{2,2\}$. That is, one partition each of 0, 1, 3, and 4 and two partitions of the integer 2. This agrees with the algebraic definition above that tells us $\binom{4}{2}_q$ should be $1 + q + 2q^2 + q^3 + q^4$.

Equivalently, we can view Figure 1 as showing the six lattice paths from $(0,0)$ to $(2,2)$. The interpretation that $\binom{n}{k}$ counts the lattice paths from $(0,0)$ to $(k, n-k)$ is often a useful one. Furthermore, it generalizes to an interpretation for $\binom{n}{k}_q$. For each lattice path, we look at the region above and to the left of the path (i.e., the shaded region in each component in Figure 1). If the area of that region is i, then the lattice path has weight q^i. $\binom{n}{k}_q$ is the sum of the weights of all lattice paths from $(0,0)$ to $(k, n-k)$.

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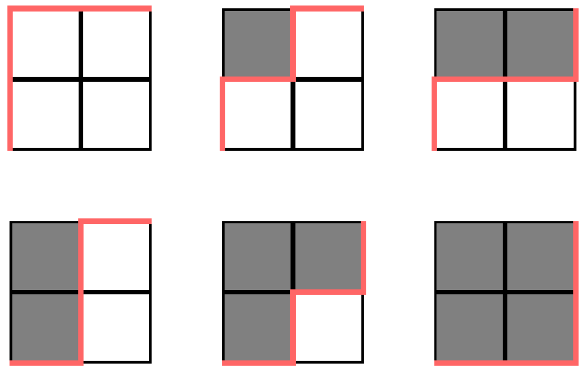

Figure 1. The six integer partitions that fit into a 2×2 box

Yet another option is to view $\binom{n}{k}$ as counting the number of ways to tile a board of size $1 \times n$ with tiles of size 1×1, of which k are blue and $n-k$ are red. To generalize this interpretation to one that works for the q-binomial coefficients, we describe a weight-preserving bijective mapping between lattice paths and board tilings.

The bijection we use is the standard one. Walking along the lattice path from $(0,0)$ to $(k, n-k)$, place a tile on the board for each step in the path—a red tile for each unit step right and a blue tile for each step up. We also assign a weight to tilings in such a way that each tiling's weight is the same as the weight of the corresponding lattice path. Let $\mathcal{T}_{n,k}$ be the set of all tilings of an n-board using exactly k blue squares and $n-k$ red squares. To each tiling T, we associate a weight denoted by q^{w_T}. For each $T \in \mathcal{T}_{n,k}$, we calculate q^{w_T} using the following procedure:

(1) Assign a weight to each square in the tiling.
   (a) Each red square gets weight 1.
   (b) For each blue square, let s be the number of red squares occurring to its left in the tiling. The blue square is assigned weight q^s.
(2) The weight of the tiling, q^{w_T}, is the product of the weights of the squares.

For example, the weight of the tiling $rbrrbb$ is $1 \cdot q^1 \cdot 1 \cdot 1 \cdot q^3 \cdot q^3 = q^7$. In Figure 2, we illustrate the correspondence between this particular tiling and its lattice path. That the two weighting schema will agree can be seen by noting that the bottommost row and the leftmost blue tile must contribute the same weight to the lattice path and tiling, respectively. The same is true of the second row from the bottom and the second blue tile from the left, and so on.

The above procedure allows us to find the weight q^{w_T} of a single tiling T. The q-binomial coefficient $\binom{n}{k}_q$ is then given by summing up the weights of all distinct tilings of a board of length n using k blue squares and $n-k$ red squares. That is,

$$\binom{n}{k}_q = \sum_{T \in \mathcal{T}_{n,k}} q^{w_T}. \tag{1.2}$$

Just as we have previously discussed the weight of an individual tile and the weight of a tiling, we will extend our terminology and say that $\binom{n}{k}_q$ gives the weight of the set of tilings $\mathcal{T}_{n,k}$.

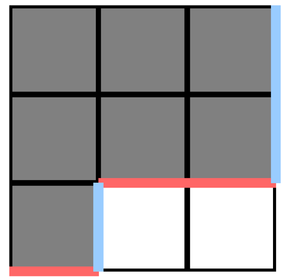

FIGURE 2. The lattice path associated with the board tiling *rbrrbb*.

In general, for any collection $\mathcal{A}$ of tilings, we use $|\mathcal{A}|$ to denote the weight of that collection. That is,

$$|\mathcal{A}| = \sum_{A \in \mathcal{A}} q^{w_A}. \tag{1.3}$$

2. Proofs of Well-Known q-Identities

In this section, we present q-analogues of several common binomial identities. The identities in this section are all presented in Andrews and Eriksson [2], with the exception of the q-Vandermonde convolution, which appears in Gould and Srivastava [6] among other sources. Andrews and Eriksson provide bijective combinatorial proofs for many of these identities using the partition-in-a-box interpretation. We present them here with proofs that make use of our tiling interpretation.

Identity 1. *For* $0 \le k \le n$,

$$\binom{n}{k}_q = \binom{n}{n-k}_q \tag{2.1}$$

Proof. Because the left side of the equation gives the weight of $\mathcal{T}_{n,k}$ and the right side gives the weight of $\mathcal{T}_{n,n-k}$, all we need is a weight-preserving bijection between those two sets. In proving the standard binomial identity,

$$\binom{n}{k} = \binom{n}{n-k} \tag{2.2}$$

we might elect to use the bijection in which a tiling of length n with k blue tiles is mapped to a tiling with $n-k$ blue tiles by toggling the color of each tile between red and blue. However, this mapping will no longer work because it is not weight-preserving. (For example, *brbrbb*, which has weight q^5 would be mapped to *rbrbrr*, which has weight q^3.)

If we instead choose the mapping in which we toggle the colors in a tiling and then reverse the tiling's order, we do indeed obtain a weight-preserving bijection. The preservation of weight is a consequence of counting the number of red tiles before each blue is equivalent to counting the number of blue tiles after each red. $\square$

Identity 2. *For* $n \ge 1$, *and* $0 \le k \le n$,

$$\binom{n}{k}_q = \binom{n-1}{k}_q + \binom{n-1}{k-1}_q q^{n-k}. \tag{2.3}$$

Proof. Consider the question of the weight of $\mathcal{T}_{n,k}$. By definition, this weight is $\binom{n}{k}_q$. However, we could also calculate this weight by separating $\mathcal{T}_{n,k}$ into two subsets, conditioned on the color of the last tile in each board. Let us call those two subsets $\mathcal{T}_{n,k}^{\text{red}}$ and $\mathcal{T}_{n,k}^{\text{blue}}$. For those boards ending in a red tile, the final red tile contributes no weight and the remainder of the board belongs to the set $\mathcal{T}_{n-1,k}$. Thus, the weight of $\mathcal{T}_{n,k}^{\text{red}}$ is $\binom{n-1}{k}_q$. For all boards in $\mathcal{T}_{n,k}^{\text{blue}}$, the final blue tile contributes weight q^{n-k} because there are $n-k$ red tiles preceding it. The remainder of the board is in $\mathcal{T}_{n-1,k-1}$, so the weight of $\mathcal{T}_{n,k}^{\text{blue}}$ is $\binom{n-1}{k-1}_q q^{n-k}$, as desired. □

Identity 3. *For $n \geq 1$, and $0 \leq k \leq n$,*

$$\binom{n}{k}_q = \binom{n-1}{k}_q q^k + \binom{n-1}{k-1}_q. \tag{2.4}$$

Proof. Follow the same procedure as in the above proof, but condition instead on the color of the first tile. Note that a leading blue tile contributes no weight, whereas a leading red tile contributes weight q^k because it has k blue tiles to its right. □

Identity 4. *For $n \geq 0$,*

$$\binom{2n}{n}_q = \sum_{j=0}^{n} q^{j^2} \binom{n}{j}_q^2. \tag{2.5}$$

Proof. Consider the weight of $\mathcal{T}_{2n,n}$. On the one hand, we know this to be $\binom{2n}{n}_q$. On the other hand, we can calculate this weight by partitioning $\mathcal{T}_{2n,n}$ into subsets conditioned on the number of red tiles out of the first n tiles. In the case where this number is j, our board as a whole consists of a board from $\mathcal{T}_{n,n-j}$ concatenated with a board from $\mathcal{T}_{n,j}$. When calculating the weight of this board, we can multiply the weights of its two halves, but we must include a factor to account for the second half of the board having extra tiles to its left. Because the left half of the board contains j red tiles, each blue tile in the right half gets additional weight q^j. Furthermore, because there are j blue tiles in the right half, the total weight contributed by the concatenation is q^{j^2}.

Thus,

$$\binom{2n}{n}_q = \sum_{j=0}^{n} q^{j^2} \binom{n}{j}_q \binom{n}{n-j}_q. \tag{2.6}$$

And applying Identity 1 gives the desired result. □

Identity 5 (The q-Binomial Theorem)**.** *For $n \geq 1$,*

$$\prod_{j=0}^{n-1} (1 + xq^j) = \sum_{k=0}^{n} q^{\binom{k}{2}} \binom{n}{k}_q x^k. \tag{2.7}$$

Proof. First, we provide a proof of the standard binomial theorem using generating functions, as our proof of the q-version will follow along the same lines.

Lemma 2.1 (The Binomial Theorem)**.** *For $n \geq 0$,*

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \tag{2.8}$$

Proof. To prove this lemma, we consider a combinatorial interpretation of $(1 + x)^n$ treated as a generating function in x. One such interpretation is tiling a board of length n with red and blue tiles. Each multiple of $(1+x)$ represents an independent choice between a red tile and a blue tile, with the exponent on x keeping track of the total number of blue tiles selected. As a result, if we let a_k count the number of distinct boards of length n that contain precisely k blue tiles, then our generating function must look like

$$\sum_k a_k x^k. \tag{2.9}$$

But a_k, as we defined it, is clearly just $\binom{n}{k}$, meaning the left and right sides of (2.8) are generating functions for the same combinatorial object, completing the proof. □

Returning to the q-binomial theorem, we will now attempt to construct a generating function that generalizes (2.8). Start again with an interpretation in which we are building up boards of length n by making n independent choices of tile color. Because we now must keep track of the weight each tile contributes, terms in the product look like $(1+xq^i)$. The exponent i would ideally represent the weight contributed by the corresponding blue tile. Because the correct value of i actually depends on the number of blue tiles already selected, we make a simplifying approximation and correct for the error this introduces later. The approximation we make is that a blue tile on position i contributes weight q^{i-1}. (Note that this is precisely correct only for the first blue tile. All subsequent blue tiles' weights will be overstated by some factor.) Thus, we start with a generating function of

$$\prod_{j=1}^{n}(1+xq^{j-1}), \tag{2.10}$$

recognizing in advance that most tilings' weights will not match up with their weights as calculated by $\binom{n}{k}_q$.

So, what precisely is the coefficient on x^k of this generating function? As long as the error introduced by our simplification above depends only on k, then our generating function will be

$$\sum_{k=0}^{n} q^{e(k)}\binom{n}{k}_q x^k, \tag{2.11}$$

where $q^{e(k)}$ is the error term due to overcounting the weight of each blue tile. Thankfully, this error term does depend only on k. As noted above, the weight of the first blue tile, according to our generating function, agrees with the weight of the first blue tile as calculated in $\binom{n}{k}_q$. However, suppose the second blue tile selected comes at position j. Then our generating function assigns it weight q^{j-1}, whereas it should have received a weight of q^{j-2} (because it has only $j-2$ red squares to its left). Similarly, the third blue square is assigned a weight that is too high by a factor of q^2, and so on. Thus, we can calculate the total error term

$$e(k) = 0+1+2+\cdots+(k-1) = \binom{k}{2}. \tag{2.12}$$

This yields

$$\prod_{j=1}^{n}(1+xq^{j-1}) = \sum_{k=0}^{n} q^{\binom{k}{2}}\binom{n}{k}_q x^k, \tag{2.13}$$

from which point a change of indices yields the q-binomial theorem. □

Identity 6 (The q-Vandermonde Convolution)**.** *For any* $p \in \{0, 1, 2, \dots, n\}$,

$$\binom{n}{m}_q = \sum_k \binom{p}{k}_q \binom{n-p}{m-k}_q q^{(p-k)(m-k)}. \tag{2.14}$$

Proof. Consider the question of the weight of $\mathcal{T}_{n,m}$. By definition, this weight is $\binom{n}{m}_q$. Now, fix a value of p in $\{0, 1, \dots, n\}$ and partition the tilings in $\mathcal{T}_{n,m}$ according to how many of the first p tiles are blue. (Call this number k.) For a fixed k, the sum of the weights of the tilings of the first p positions is $\binom{p}{k}_q$. If we ignore the first p positions, the sum of the weights of the tilings on the last $n-p$ positions is $\binom{n-p}{m-k}_q$. However, we must additionally adjust this weight because of the concatenation. Because there are $p-k$ red tiles in the first section, each blue tile in the second section should get additional weight q^{p-k}. Combining the additional weights from the $m-k$ blue tiles in the second section gives a total extra weight of $q^{(p-k)(m-k)}$ on the second section due to the red tiles in the first section. Therefore, the total weight of the tilings we are looking for is

$$\binom{n}{m}_q = \sum_k \binom{p}{k}_q \binom{n-p}{m-k}_q q^{(p-k)(m-k)}, \tag{2.15}$$

as desired. □

If we instead partition on the number of blue tilings in the *last* p positions, we get the similar identity

$$\binom{n}{m}_q = \sum_k \binom{p}{k}_q \binom{n-p}{m-k}_q q^{k(n-p-m+k)}. \tag{2.16}$$

3. Advanced q-identities

3.1. **Sums of Consecutive Integers, Squares, and Cubes.**

The binomial identity

$$\sum_{k=1}^{n} k = \binom{n+1}{2} \tag{3.1}$$

can be proved by considering the location of the last blue square in a tiling of length $n+1$ with exactly two blue squares. We prove the following q-analogue using the same interpretation.

Identity 7 (Sum of q-integers)**.** *For* $n \geq 1$,

$$\sum_{k=1}^{n} q^{k-1} k_q = \binom{n+1}{2}_q. \tag{3.2}$$

Proof. Consider the weight of $\mathcal{T}_{n+1,2}$. On the one hand, we know this to be $\binom{n+1}{2}_q$. On the other hand, each tiling in $\mathcal{T}_{n+1,2}$ has exactly two blue squares. Consider the location of the second blue square. Label the locations of the board from 1 to $n+1$ and call the position of the second blue square $k+1$. Then the second blue square has $k-1$ red squares before it and therefore, has weight q^{k-1}. The first blue square, on the other hand, can be preceded by 0 or 1 or 2 or … or $k-1$ red squares. Thus, the total weight of the tilings with the second blue square in position $k+1$ is $q^{k-1}(1 + q + \cdots + q^{k-1}) = q^{k-1} k_q$. The weight of $\mathcal{T}_{n+1,2}$ is given by summing $q^{k-1} k_q$ over all possible locations $k+1$ of the second blue tile. That is,

$$\binom{n+1}{2}_q = \sum_{k+1=2}^{n+1} q^{k-1} k_q = \sum_{k=1}^{n} q^{k-1} k_q, \tag{3.3}$$

as claimed. □

We proceed to an identity on the sum of integer *cubes*. (Identities on the sum of integer squares turn out to be slightly more complicated and will be addressed after the cubes.)

Garrett and Hummel [5] provide the following identity on the sum of q-cubes.

$$\sum_{k=1}^{n} q^{k-1}\left(\frac{1-q^k}{1-q}\right)^2\left(\frac{1-q^{k-1}}{1-q^2}+\frac{1-q^{k+1}}{1-q^2}\right)=\binom{n+1}{2}_q^2. \tag{3.4}$$

We provide a novel variant of that identity.

Identity 8 (Sum of q-cubes)**.** *For $n \geq 1$,*

$$\sum_{k=1}^{n} q^{k-1}k_q^3=\binom{n+1}{2}_q^2+(q-1)\sum_{k=1}^{n} q^{k-1}k_q\binom{k}{2}_q. \tag{3.5}$$

Note that this identity has the nice property of being immediately recognizable as a generalization of the standard binomial identity for the sum of cubes:

$$\sum_{k=1}^{n} k^3=\binom{n+1}{2}^2. \tag{3.6}$$

If we take $q = 1$ in (3.5), the final term drops out entirely and from the remaining terms we obtain (3.6).

Proof. To avoid the hard-to-interpret subtraction present in the factor of $(q-1)$, we provide a combinatorial proof for the slightly rearranged identity,

$$\sum_{k=1}^{n} q^{k-1}k_q^3+\sum_{k=1}^{n} q^{k-1}k_q\binom{k}{2}_q=\binom{n+1}{2}_q^2+q\sum_{k=1}^{n} q^{k-1}k_q\binom{k}{2}_q. \tag{3.7}$$

Before doing so, it will be helpful to introduce some new notation. Let $(x_1, x_2, \ldots, x_C)_n$ denote the weighted tiling of a board of length n with blue squares at positions x_1, x_2, …, x_C and red squares elsewhere. We require $1 \leq x_1 < x_2 < \cdots < x_C \leq n$. We can calculate the weight of this tiling as

$$q^{\sum_{i=1}^{C} x_i - i}. \tag{3.8}$$

For example, when $C = 1$, $(a)_n$ would denote a length-n tiling of weight q^{a-1} with a single blue square in the ath position. Figure 3 illustrates this new notation.

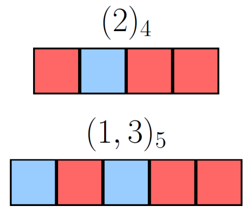

FIGURE 3. Illustration of the use of the $(x_1, x_2, \ldots, x_C)_n$ notation to describe boards.

Now, we define four sets for which the four terms in Equation (3.7) give the weight.

Let

$$\mathcal{A} = \{(a)_{n+1}, (b)_{n+1}, (c,d)_{n+1} | a, b < d\} \tag{3.9}$$

$$\mathcal{B} = \{(e,f)_{n+1}, (g,h)_{n+1} | f < h\} \tag{3.10}$$

$$\mathcal{C} = \{(i,j)_{n+1}, (l,m)_{n+1}\} \tag{3.11}$$

$$\mathcal{D} = \{(r)_{n+1}, (s)_{n+1}, (t,u)_{n+1} | r < s < u\}. \tag{3.12}$$

Elements in these sets are pairs or triples of boards, where all boards have length $n+1$. We take the weight of an N-tuple of boards $(B_1, \ldots, B_N)$ to be the product of their individual weights, i.e., $\prod_{k=1}^{N} q^{w_{B_k}}$.

We now check that the weights of these sets correspond to the terms in Equation (3.7).

Letting $k = d-1$, we observe that

$$|\mathcal{A}| = \sum_{k=1}^{n} q^{k-1} k_q^3. \tag{3.13}$$

Similarly, letting $k = h-1$, we get

$$|\mathcal{B}| = \sum_{k=1}^{n} q^{k-1} k_q \binom{k}{2}_q. \tag{3.14}$$

Straightforwardly,

$$|\mathcal{C}| = \binom{n+1}{2}_q^2. \tag{3.15}$$

Finally, by letting $k = u-1$,

$$|\mathcal{D}| = \sum_{k=1}^{n} q^{k-1} k_q \cdot q \binom{k}{2}_q. \tag{3.16}$$

To prove the identity, we need a weight-preserving bijection from $\mathcal{A} \cup \mathcal{B}$ to $\mathcal{C} \cup \mathcal{D}$. Note that $\mathcal{B} \subset \mathcal{C}$ and $\mathcal{D} \subset \mathcal{A}$, so most of our bijection can be accomplished with the identity map. The complete bijection is given by the following map:

- For each element of $\mathcal{B}$, map via the identity map into $\mathcal{C}$. All elements of $\mathcal{C}$ are hit except those where $j \geq m$.
- For each element of $\mathcal{A}$ with $a < b$, map via the identity map into $\mathcal{D}$. This covers all of $\mathcal{D}$.
- For the remaining elements of $\mathcal{A}$ with $a \geq b$, put the triplet $((a)_{n+1}, (b)_{n+1}, (c,d)_{n+1}) \in \mathcal{A}$ in correspondence with the doublet $((c,d)_{n+1}, (b, a+1)_{n+1}) \in \mathcal{C}$, which has the same weight and satisfies $j \geq m$.

This map is 1-1, onto, and weight preserving, so we have our identity. □

We now conclude this subsection with a pair of q-analogues to identities on sums of integer squares. Michael Schlosser [7] presents two q-analogues for identities on the sum of integer squares:

$$\sum_{k=1}^{n} \frac{(1-q^{2k})(1-q^k)}{(1-q^2)(1-q)} q^{\frac{3}{2}(n-k)} = \frac{(1-q^n)(1-q^{n+1})(1-q^{n+1/2})}{(1-q)(1-q^2)(1-q^{3/2})} \tag{3.17}$$

and

$$\sum_{k=1}^{n} \frac{(1-q^{3k})(1-q^k)}{(1-q^3)(1-q)} q^{2(n-k)} = \frac{(1-q^n)(1-q^{n+1})(1-q^{2n+1})}{(1-q)(1-q^2)(1-q^3)}. \tag{3.18}$$

We present two further sum-of-squares identities:

Identity 9 (Sum of q-squares, Variant 1)**.** *For $n \geq 1$,*

$$\sum_{k=1}^{n} q^{k-1} k_q^2 = 2q^2 \binom{n+1}{3}_q + \sum_{k=1}^{n} q^{k-1} k_{q^2} \tag{3.19}$$

and

Identity 10 (Sum of q-squares, Variant 2)**.** *For $n \geq 1$,*

$$(1+2q+q^2) \sum_{k=1}^{n-1} k_{q^2}^2 q^{2k-1} = \binom{2n}{3}_q + (q-1) \sum_{k=1}^{n-1} (2k-1)_q (2k)_q q^{2k-2}. \tag{3.20}$$

As before, a nice feature of these identities is that when we set $q = 1$, we straightforwardly recover standard binomial identities, namely

$$\sum_{k=1}^{n} k^2 = 2\binom{n+1}{3} + \binom{n+1}{2} \tag{3.21}$$

and

$$4\sum_{k=1}^{n-1} k^2 = \binom{2n}{3}. \tag{3.22}$$

We now prove these two new identities.

Proof of Identity 9. Fix a positive integer n and consider the weight of the set

$$\{(a)_{n+1}, (b,c)_{n+1} | a < c\}.$$

Here, we continue to use the notation introduced in the proof of Identity 8, so this set consists of tilings of pairs of boards of length $n+1$.

One simple way to calculate the weight of this set is to condition on the value of c. If $c = k+1$, then the blue tile in position c has weight $q^{k+1-2} = q^{k-1}$. Having picked c, we can then independently choose a and b to be anything from 1 to k. The weight of the tile on a will be one of the k terms in the polynomial k_q. Likewise, the weight of the tile on position b can also be any of the k terms in k_q. Together, the sum of the weights of all tilings in the set is

$$\sum_{k=1}^{n} q^{k-1} k_q^2.$$

On the other hand, we might split the weight calculation up into cases, depending on whether or not $a = b$. If $a < b$, then we can create the the paired tiling $(a)_{n+1}, (b,c)_{n+1}$ by taking the tiling $(a,b,c)_{n+1}$ and breaking out the blue tile on position a onto a separate board. Thus, the number of such tilings with $a < b$ is $\binom{n+1}{3}$. Furthermore, the weight of $(a)_{n+1}, (b,c)_{n+1}$ is greater than the corresponding tiling $(a,b,c)_{n+1}$ by a factor of q^2. Thus, the sum of the weights of the tilings with $a < b$ is

$$q^2 \binom{n+1}{3}_q.$$

Likewise, the sum of the weights of the tilings with $a > b$ is also

$$q^2\binom{n+1}{3}_q.$$

Finally, we consider the case where $a = b$. In this case, we again condition on the value of c. If $c = k+1$, then the tile on position c gets weight q^{k-1}. We may then choose the values of a and b, remembering our constraint that $a = b$. If we take $a = b = 1$, then the tiles on a and b have a combined weight of 1. If we take $a = b = 2$, they have a combined weight of q^2. In general, if $a = b = i$, those two tiles contribute a weight of q^{2i-2}. Thus, the weights of the tilings with $c = k+1$ and $a = b$ are given by

$$q^{k-1}(1 + q^2 + q^4 + \cdots + q^{2k-2}) = q^{k-1}k_{q^2}.$$

Combining these two cases together, the total weight of the tilings in the set is given by

$$2q^2\binom{n+1}{3}_q + \sum_{k=1}^{n} q^{k-1}k_{q^2},$$

as desired. □

Proof of Identity 10. The goal is to prove that

$$(1+2q+q^2)\sum_{k=1}^{n-1} k_{q^2}^2 q^{2k-1} = \binom{2n}{3}_q + (q-1)\sum_{k=1}^{n-1}(2k-1)_q(2k)_q q^{2k-2}. \tag{3.23}$$

Instead, we prove the slightly rearranged identity:

$$(1+2q+q^2)\sum_{k=1}^{n-1} k_{q^2}^2 q^{2k-1} + \sum_{k=1}^{n-1}(2k-1)_q(2k)_q q^{2k-2} = \binom{2n}{3}_q + q\sum_{k=1}^{n-1}(2k-1)_q(2k)_q q^{2k-2}. \tag{3.24}$$

Our proof will consist of two steps. First, we will define sets whose weights are given by each of the four terms in (3.24). Secondly, we will give a weight-preserving bijection between the sets on the left side and the right side.

Let

$$\begin{aligned}
\mathcal{A} &= \{(a)_{2n}, (b,c)_{2n} | a < c,\ c \text{ odd}\}\\
\mathcal{B} &= \{(e,f,g)_{2n}\}\\
\mathcal{C} &= \{(r)_{2n}, (s,t)_{2n} | t \le 2n-2,\ r \le t,\ t \text{ even}\}\\
\mathcal{D} &= \{(u)_{2n}, (v,w)_{2n} | u < w-1,\ w \text{ odd}\}.
\end{aligned}$$

What are the weights of these sets? Note that

$$\sum_{k=1}^{n-1} k_{q^2}^2 q^{2k-1}$$

gives the weight of the tilings in the set $\{(a)_{2n}, (b,c)_{2n} | a, b < c \text{ and } a, b, c \text{ odd}\} =: \mathcal{A}_{odd,odd}$, which is a subset of $\mathcal{A}$. This set, $\mathcal{A}_{odd,odd}$, includes only one-fourth of the tilings in $\mathcal{A}$, namely, those in which both a and b are odd. To get the full weight of $\mathcal{A}$, we need also consider $\mathcal{A}_{odd,even}$, $\mathcal{A}_{even,odd}$, and $\mathcal{A}_{even,even}$.

What about the tilings with a odd and b even? We can construct these tilings easily from those in $A_{odd,odd}$ by increasing each value of b by one. This increases the weight of each tiling by a factor of q, so

$$|\mathcal{A}_{odd,even}| = q|\mathcal{A}_{odd,odd}|.$$

Likewise,

$$|\mathcal{A}_{even,odd}| = q|\mathcal{A}_{odd,odd}|$$

and

$$|\mathcal{A}_{even,even}| = q^2|\mathcal{A}_{odd,odd}|,$$

so

$$|\mathcal{A}| = (1+2q+q^2)|\mathcal{A}_{odd,odd}| = (1+2q+q^2)\sum_{k=1}^{n-1} k_{q^2}^2 q^{2k-1}.$$

Secondly, by our tiling interpretation of the q-binomial coefficient, the weight of the collection $\mathcal{B}$ is given by

$$|\mathcal{B}| = \binom{2n}{3}_q.$$

Thirdly, by letting $t = 2k$, the sum of the weights of the tilings in $\mathcal{C}$ is given by

$$|\mathcal{C}| = \sum_{k=1}^{n-1}(2k-1)_q(2k)_q q^{2k-2}.$$

Finally, by letting $w = 2k+1$, the sum of the weights of the tilings in $\mathcal{D}$ is given by

$$|\mathcal{D}| = \sum_{k=1}^{n-1}(2k-1)_q(2k)_q q^{2k-1}.$$

Now, we must define a weight-preserving bijection from $\mathcal{A}\cup\mathcal{C}$ to $\mathcal{B}\cup\mathcal{D}$.

Because $\mathcal{D}\subset\mathcal{A}$, it makes sense to rewrite $\mathcal{A}$ as $\mathcal{E}\cup\mathcal{F}$, where $\mathcal{E}=\mathcal{D}$ and $\mathcal{F}$ consists of those elements in $\mathcal{A}$ that satisfy the additional restriction that $a = c-1$.

The weight-preserving bijection we use is then:

$$\mathcal{E}\to\mathcal{D}$$

$$(c-1)_{2n},(b,c)_{2n}\in\mathcal{F}\to(b,c,c+1)_{2n}\in\mathcal{B}$$

$$(r)_{2n},(s,t)_{2n}\in\mathcal{C}\to\begin{cases}(s,r,t+2)_{2n}\in\mathcal{B}, & \text{if } r>s;\\ (r,r+1,t+1)_{2n}\in\mathcal{B}, & \text{if } r=s;\\ (r,s+1,t+1)_{2n}\in\mathcal{B}, & \text{if } r<s.\end{cases}$$

It is then easy to check that this bijection is one-to-one, onto, and weight-preserving, which completes our proof. □

3.2. **An Alternating Sum Identity.** In Andrews and Eriksson [2], the following identity is proven by algebraic substitution of the recurrence relation (2.3). We present instead a combinatorial proof using tilings.

Identity 11 (The Gaussian Formula)**.** *For $n \geq 1$,*

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}_q = \begin{cases}0, & \text{if } n \text{ is odd};\\ (1-q)(1-q^3)(1-q^5)\cdots(1-q^{n-1}), & \text{if } n \text{ is even}.\end{cases} \tag{3.25}$$

Proof. Let $\mathcal{T}_n$ be the collection of all tilings of length n. Define $\mathcal{T}_{n,odd}$ and $\mathcal{T}_{n,even}$ to be the tilings of length n with an even number of blue tiles and an odd number of blue tiles, respectively.

For odd n, we need only show that $|\mathcal{T}_{n,odd}| = |\mathcal{T}_{n,even}|$. This we can accomplish by demonstrating a weight-preserving bijection between $\mathcal{T}_{n,odd}$ and $\mathcal{T}_{n,even}$. For each tiling $T\in\mathcal{T}_{n,odd}$, create the associated tiling $T'\in\mathcal{T}_{n,even}$ by reversing the order of T and toggling the color of each tile. This procedure preserves the weight of T, but changes the parity of the number of blue tiles. Furthermore, if we perform the same procedure on T', we will get T back. Hence, $|\mathcal{T}_{n,odd}| = |\mathcal{T}_{n,even}|$, as claimed.

The case where n is even is more complicated. If we can show combinatorially that

$$\sum_{j=0}^{n+2}(-1)^j\binom{n+2}{j}_q = (1-q^{n+1})\sum_{j=0}^{n}(-1)^j\binom{n}{j}_q, \tag{3.26}$$

then we may proceed by induction. To show that this equality holds, partition the collection $\mathcal{T}_{n+2}$ into four distinct cases. For each tiling $T \in \mathcal{T}_{n+2}$, exactly one of the following must hold:

Case 1: T begins with bb.
Case 2: T begins with br.
Case 3: T begins with r and ends with b.
Case 4: T begins with r and ends with r.

Each of these cases represents precisely one-fourth of the tilings. Cases 2 and 4 cancel each other out in the alternating sum. Note that a b at the beginning of a tiling adds no weight to a tiling, nor does an r at the end. Thus, for each tiling in case 2, we can change the leading b to a trailing r and obtain a tiling in case 4 with equal weight but opposite parity of the number of blue tiles. Thus,

$$\begin{aligned}\sum_{j=0}^{n+2}(-1)^j\binom{n+2}{j}_q = &\sum_{\substack{T\in \text{ Case 1,}\\ \text{\# of blue tiles is even}}} q^{w_T} - \sum_{\substack{T\in \text{ Case 1,}\\ \text{\# of blue tiles is odd}}} q^{w_T}\\ &+ \sum_{\substack{T\in \text{ Case 3,}\\ \text{\# of blue tiles is even}}} q^{w_T} - \sum_{\substack{T\in \text{ Case 3,}\\ \text{\# of blue tiles is odd}}} q^{w_T}.\end{aligned} \tag{3.27}$$

Now, all tilings in case 1 begin with bb. Note that adding bb to the beginning of a tiling of length n changes neither its weight nor its parity, so

$$\sum_{\substack{T\in \text{ Case 1,}\\ \text{\# of blue tiles is even}}} q^{w_T} - \sum_{\substack{T\in \text{ Case 1,}\\ \text{\# of blue tiles is odd}}} q^{w_T} = \sum_{j=0}^{n}(-1)^j\binom{n}{j}_q. \tag{3.28}$$

What about the tilings in case 3? All such tilings are of length $n+2$, begin with r, and end with b. We can construct these from tilings of length n by first adding a leading r and then a trailing b. Assume we have a tiling T_0 of length n with j blue squares and $n-j$ red squares. Then, rT_0 will have weight $q^j q^{w_{T_0}}$, because the leading r increases the weight of each blue square by 1. Adding a trailing b, we see that the final b has weight $n-j+1$. Thus, the tiling rT_0b has weight $q^j q^{w_{T_0}} q^{n-j+1} = q^{w_{T_0}} q^{n+1}$. Furthermore, adding one red and one blue square has changed the parity of the number of blue squares, so

$$\begin{aligned}\sum_{\substack{T\in \text{ Case 3,}\\ \text{\# of blue tiles is even}}} q^{w_T} - \sum_{\substack{T\in \text{ Case 3,}\\ \text{\# of blue tiles is odd}}} q^{w_T} &= \sum_{j=0}^{n} q^{n+1}(-1)^{j+1}\binom{n}{j}_q\\ &= -q^{n+1}\sum_{j=0}^{n}(-1)^j\binom{n}{j}_q,\end{aligned} \tag{3.29}$$

and therefore,

$$\sum_{j=0}^{n+2}(-1)^j\binom{n+2}{j}_q = (1-q^{n+1})\sum_{j=0}^{n}(-1)^j\binom{n}{j}_q, \tag{3.30}$$

as claimed.

Thus, since

$$\sum_{j=0}^{2}(-1)^j\binom{2}{j}_q = \binom{2}{0}_q - \binom{2}{1}_q + \binom{2}{2}_q = 1-(1+q)+1 = 1-q, \tag{3.31}$$

by induction we have

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}_q = (1-q)(1-q^3)(1-q^5)\cdots(1-q^{n-1})$$

for even values of n, completing the proof. □

3.3. **A q-multichoose Identity.** One standard combinatorial definition of "multichoose" is that $\left(\!\binom{n}{k}\!\right)$ counts the number of distinct ways to distribute k identical balls into a row of n labeled urns. It is easy to show from this definition that $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$. Such a definition extends naturally to a board-tiling interpretation. The basic idea is that $\left(\!\binom{n}{k}\!\right)$ counts the number of tilings of a board of length n using $n-1$ red tiles and k blue tiles, where tile stacking is allowed. Start with an empty board of length n, with positions labeled from 0 through $n-1$. The locations of the $n-1$ red tiles are all predetermined. Place one red tile at the bottom of the stacks at locations 1 through $n-1$. The remaining k blue tiles may each be placed on top of any of the existing stacks in locations 0 through $n-1$. There is no restriction on how many tiles are allowed in a stack.

Note, in particular, that the stacks at locations 1 through $n-1$ will always have at least one tile, because we started by placing red tiles there. The stack at location 0, on the other hand, has no red tiles and could possibly end up with no tiles at all.

If we now give a method for calculating the weight of a stacked tiling, then we will have a q-analogue of multichoose. The method we choose is that each red tile gets weight 1 and each blue tile stacked on position i gets weight q^i. The weight of the tiling is then the product of the weights of the individual tiles. Equivalently, we may calculate the weight of such a stacked tiling by collapsing down to a standard, unstacked board tiling and calculating the weight in the usual manner. To convert from a stacked to an unstacked tiling, we simply work through the stacked tiling from left to right, toppling any vertical stacks to create instead horizontal rows. Figure 4 gives an example of such a stacked tiling and its corresponding unstacked tiling.

The following identity is taken from Andrews and Eriksson [2], who prove it combinatorially using a partition-in-a-box interpretation. Here, we prove it instead with a stackable tiling interpretation.

Identity 12 (The q-binomial series)**.** *For $n \geq 1$,*

$$\prod_{j=1}^{n}\frac{1}{1-zq^j} = \sum_{m=0}^{\infty} q^m \left(\!\binom{n}{m}\!\right)_q z^m. \tag{3.32}$$

Proof. We prove instead the equivalent identity

$$\prod_{j=0}^{n-1}\frac{1}{1-zq^j} = \sum_{m=0}^{\infty} \left(\!\binom{n}{m}\!\right)_q z^m, \tag{3.33}$$

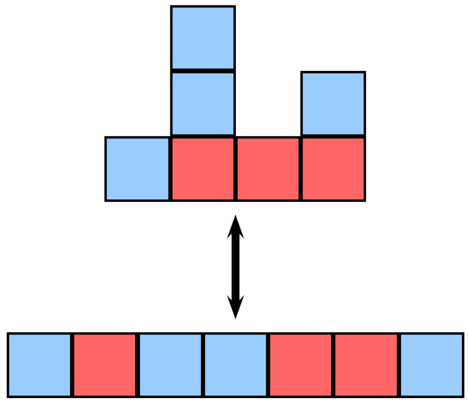

FIGURE 4. A stacked tiling of weight q^5 and its corresponding unstacked tiling, also of weight q^5.

where the limits of the product on the left side have been shifted and the q^m has disappeared from the right side. To see that these two identities are equivalent, replace each occurrence of z in (3.33) by qz to obtain (3.32).

Here, the most straightforward approach is to use generating functions. Suppose we want to determine a generating function in z for the weighted number of stackable tilings on positions 0 through $n-1$, where the power on z counts the total number of blue tiles used. By definition, this generating function is

$$\sum_{m=0}^{\infty} \left(\!\binom{n}{m}\!\right)_q z^m.$$

On the other hand, we can construct such a generating function by, in turn, considering how many blue tiles are placed at each position. At each of the n positions, we may independently choose any nonnegative number of blue tiles. At position 0, each blue tile contributes a factor of zq^0, so our generating function should have a factor of $(1+z+z^2+z^3+\cdots)$. Next, at position 1, we may independently choose to have any nonnegative number of green tiles, each of which contributes weight of zq^1, so our generating function has a factor of $(1+zq+(zq)^2+(zq)^3+\cdots)$. Continuing this pattern up to position $n-1$, our overall generating function is

$$(1+z+z^2+\cdots)\cdots(1+zq^{n-1}+(zq^{n-1})^2+\cdots) = \prod_{j=0}^{n-1} \frac{1}{1-zq^j}.$$

Thus, because both expressions provide generating functions for the same collection, it follows that

$$\prod_{j=0}^{n-1} \frac{1}{1-zq^j} = \sum_{m=0}^{\infty} \left(\!\binom{n}{m}\!\right)_q z^m,$$

as claimed. □

3.4. **Palindromic Tilings.** We might ask ourselves whether palindromic tilings have any interesting properties, and in particular, whether collections of palindromic tilings have nicely expressible weights. We consider two different types of palindromic tilings. Firstly, we examine tilings that are the same when reversed, and secondly, tilings that are unchanged when we toggle all the tiles between red and blue and then reverse the order. We will call these palindromic tilings of the first and second type, respectively. Figure 5 illustrates some examples of palindromic tilings of both types.

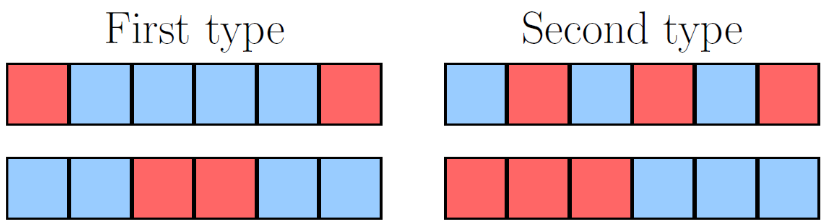

FIGURE 5. Example palindromic tilings of the first and second type.

For notational convenience, we will denote the collection of type-i palindromic tilings of boards of length n using k blue squares and $n-k$ red squares as $\mathcal{P}^i_{n,k}$. Let us first examine the collection of type-1 palindromic tilings of even length, with a fixed number of blue. In our new notation, we denote this collection by $\mathcal{P}^1_{2n,2k}$. (Note that in even-length, type-1 palindromic tilings, every square in a tiling is matched by another square of the same color on the opposite side of the board, so the number of blue squares is guaranteed to be even.) We now set out to find the weight of this collection, $|\mathcal{P}^1_{2n,2k}|$. First, we provide a recurrence relation for the weight of this collection, and then a closed form solution.

Identity 13. *For $n \geq 1$, $0 \leq k \leq n$,*

$$|\mathcal{P}^1_{2n,2k}| = |\mathcal{P}^1_{2n-2,2k-2}| \cdot q^{2(n-k)} + |\mathcal{P}^1_{2n-2,2k}| \cdot q^{2k}. \tag{3.34}$$

Proof. Consider the color of the first square in a tiling $T \in \mathcal{P}^1_{2n,2k}$. Suppose this first square is blue (and therefore, the last square in T is blue as well). Then, T has the form $bT'b$, where T' is a palindromic tiling of length $2n-2$ with $2k-2$ blue squares. Now, the leading blue square will contribute no weight to T, whereas the trailing blue square will contribute weight $q^{2(n-k)}$ from the $2(n-k)$ red squares preceding it. That is, $q^{w_T} = q^{2(n-k)}q^{w_{T'}}$. Thus, the sum of the weights of the tilings in $\mathcal{P}^1_{2n,2k}$ that begin with a blue square is $|\mathcal{P}^1_{2n-2,2k-2}| \cdot q^{2(n-k)}$.

Likewise, for those tilings that begin and end with a red square, our complete tiling T takes the form $rT'r$. The trailing red contributes no weight and the leading red contributes an extra weight of q^{2k} to the interior palindromic tiling T'. Hence, the sum of the weights of the tilings in $\mathcal{P}^1_{2n,2k}$ that begin with a red square is $|\mathcal{P}^1_{2n-2,2k}| \cdot q^{2k}$.

Altogether, this gives the recurrence relation

$$|\mathcal{P}^1_{2n,2k}| = |\mathcal{P}^1_{2n-2,2k-2}| \cdot q^{2(n-k)} + |\mathcal{P}^1_{2n-2,2k}| \cdot q^{2k}.$$

□

Combined with appropriate base cases, it is easy to check that this recurrence relation has solution

$$|\mathcal{P}^1_{2n,2k}| = \binom{n}{k} q^{2k(n-k)}. \tag{3.35}$$

We now provide a direct combinatorial proof for Equation (3.35).

Combinatorial proof of (3.35). Our goal is to sum the weights of all palindromic tilings of the first type of length $2n$ that use $2k$ blue squares. It is natural to think of those $2k$ blue squares as k *pairs* of blue squares, with a blue square on position i matched by its mirror image on position $(2n-i+1)$. Analogously, the $2(n-k)$ red squares come in the form of $n-k$ pairs of red squares.

Now, we claim that each pair of red tiles contributes a weight of q^2 to each pair of blue tiles, regardless of the exact positions of those pairs of tiles. Why? Consider just a pair of blue tiles and a pair of red tiles in a type-1 palindromic tiling, ignoring the rest of the tiling. Because the tiling is palindromic, those blues and reds must be arranged as either *brrb* or *rbbr*. (Any intermediate tiles can be ignored for the moment because we are only considering the contribution of this one pair of reds to this one pair of blues.) If the arrangement is *brrb*, then the red tiles each contribute a weight of q to the second blue, for a total contribution of q^2. On the other hand, if the arrangement is *rbbr*, then the initial red contributes a weight of q to each of the blues, for a total contribution of q^2.

Thus, because there are $n-k$ pairs of red squares, each pair of blue squares has total weight $(q^2)^{n-k} = q^{2(n-k)}$. Furthermore, because there are k pairs of blue squares, the total weight of each tiling in $\mathcal{P}^1_{2n,2k}$ is just $q^{2k(n-k)}$. All the tilings in this collection have precisely the same weight! To find the weight of the collection, then, we need only count the number of such tilings. To fully describe a tiling in this collection, it is sufficient to give the locations of the k blue squares among the first n positions; the second half of the board will be forced to mirror the first half. Thus, the collection $\mathcal{P}^1_{2n,2k}$ contains $\binom{n}{k}$ tilings, each with weight $q^{2k(n-k)}$ and therefore,

$$|\mathcal{P}^1_{2n,2k}| = \binom{n}{k} q^{2k(n-k)},$$

as claimed. □

Until now, we have only considered collections of *even*-length, type-1 palindromic tilings. The following two identities give the weights of collections of *odd*-length, type-1 palindromic tilings.

Identity 14. *For $n \geq 1$, $0 \leq k \leq n$,*

$$|\mathcal{P}^1_{2n+1,2k}| = q^k |\mathcal{P}^1_{2n,2k}| = \binom{n}{k} q^{2k(n-k)} q^k, \tag{3.36}$$

and

Identity 15. *For $n \geq 1$, $0 \leq k \leq n$,*

$$|\mathcal{P}^1_{2n+1,2k+1}| = q^{n-k} |\mathcal{P}^1_{2n,2k}| = \binom{n}{k} q^{2k(n-k)} q^{n-k}. \tag{3.37}$$

We provide a proof for Identity 14. The proof of Identity 15 is analogous.

Proof of Identity 14. The collection $\mathcal{P}^1_{2n+1,2k}$ contains the palindromic tilings of the first kind with $2k$ blue squares and $2n-2k+1$ red squares. Because there is an odd number of red squares, the center square must be red. We can create any such tiling by taking a tiling from $\mathcal{P}^1_{2n,2k}$ and inserting a red square in the middle. Adding a red square to the middle position in such a tiling contributes an extra weight of q to each blue square in the latter half of the tiling. Thus, the insertion of the central square always adds a weight of q^k to the initial, length $2n$ tiling. From this, it follows that

$$|\mathcal{P}^1_{2n+1,2k}| = q^k |\mathcal{P}^1_{2n,2k}|,$$

and the second equality in (3.36) follows from Equation (3.35). □

Finally, what about the palindromic tilings of the *second* type? There are no type-2 palindromic tilings of *odd* length, because the single central square would need to retain its original color through the procedure of having its color toggled. Additionally, we note that half the tiles in any type-2 palindromic tiling must be blue and half must be red. Hence, when considering collections of type-2 palindromic tilings, it is sufficiently general to denote these collections by $\mathcal{P}^2_{2n,n}$. We now state and prove an identity on the weights of collections of type-2 palindromic tilings.

Identity 16. *For $n \geq 1$,*

$$|\mathcal{P}^2_{2n,n}| = (1+q)(1+q^3)(1+q^5)\cdots(1+q^{2n-1}). \tag{3.38}$$

Proof. We prove this constructively. Each palindromic tiling of the second type can be created by successive additions of tiles to the beginning and end of a previous type-2 palindromic tiling. To create a palindromic tiling of length $2n$, start with the empty board and successively add either 1) a leading b and a trailing r or 2) a leading r and a trailing b, making a total of n such additions. Adding a leading b and a trailing r contributes nothing to the weight of the tiling. On the other hand, adding a leading r and a trailing b to a tiling of length $2i$ (with i red squares and i blue squares) contributes weight q^{2i+1}. Hence, the first of the n additions gives the tiling a weight of 1 or q. The second addition contributes weight of 1 or q^3. Continue in this fashion until the last addition, which contributes a weight of 1 or q^{2n-1}. Because these choices are each made independently of the previous choices, the total weight of $\mathcal{P}^2_{2n,n}$ is the product $(1+q)(1+q^3)(1+q^5)\cdots(1+q^{2n-1})$, as claimed. □

3.5. ***q*-Lucas' Theorem.** Lucas' Theorem allows us to simplify binomial coefficients modulo a prime. Let p be a prime and a and b be nonnegative integers with $0 \leq a, b < p$. Then Lucas' Theorem says

$$\binom{pn+a}{pk+b} \equiv \binom{n}{k}\binom{a}{b} \pmod{p}. \tag{3.39}$$

We first provide a proof sketch in the standard binomial context based on the proof by Anderson, Benjamin, and Rouse [1] and then generalize it to a proof in the q-binomial context.

Identity 17 (The standard Lucas' Theorem)**.** *For a prime p and nonnegative a, b with $0 \leq a, b < p$, $0 \leq k \leq n$,*

$$\binom{pn+a}{pk+b} \equiv \binom{n}{k}\binom{a}{b} \pmod{p}. \tag{3.40}$$

Proof. Consider the question of how many ways (mod p) we can tile a board of dimensions $p \times n$ and a strip of dimensions $a \times 1$ using $pk+b$ blue unit squares and $p(n-k)+(a-b)$ red unit squares. The straightforward answer is

$$\binom{pn+a}{pk+b}.$$

(From the $pn+a$ total locations, choose which $pk+b$ hold blue squares.)

On the other hand, because we only care about the answer modulo p, let us group tilings into sets of size p and then only consider tilings that do not fall into one of those sets. Start by looking at the first column of the board. If not all the squares in this column are the same color, then moving the top square to the bottom of the column and shifting the rest of the squares up by one position will create a different column. Because p is prime, repeating this process $p-1$ times will create p distinct columns. Thus, whenever the first column is not all the same color, we can place it in a group of p distinct tilings of the board. Hence, we can ignore any tilings in which the first column is not all the same color.

If the first column *is* all the same color, move on to the second column and check if it is all one color. If not, we can perform the same shifting procedure to place it into a set of size p and disregard the entire set. If the first and the second columns are monochromatic, move on to the third column. Continue in this fashion until reaching the right side of the board.

Now, the only tilings of the board and the strip that have not been put into a set of size p are the ones in which each column of the board is a single color. How many such tilings are there? There are $\binom{n}{k}$ ways to tile the board. (We know that k of the columns must be blue. Choose which k columns are blue out of the n total columns.) Then there are $\binom{a}{b}$ ways to place the remaining b blue tiles onto the strip of length a. Hence, modulo p, our answer is

$$\binom{n}{k}\binom{a}{b},$$

as desired. $\square$

We now present a q-generalization of Lucas' Theorem.

Identity 18 (The q-Lucas' Theorem)**.** *For a prime p and non-negative a, b with $0 \leq a, b < p$, $0 \leq k \leq n$,*

$$\binom{pn+a}{pk+b}_q \equiv \binom{n}{k}\binom{a}{b}_q \ (mod\ p_q). \tag{3.41}$$

Note, in particular, that we have replaced the equivalence modulo an integer p from (3.40) with equivalence modulo a polynomial p_q. Equivalence modulo p_q should be interpreted as follows: We suppose $a \equiv b \pmod{p_q}$ if and only if there exists a polynomial $f(q)$ with integer coefficients such that $a - b = f(q)p_q$. One of the key observations required for the upcoming proof is that

$$q^p \equiv 1 \pmod{p_q}. \tag{3.42}$$

This holds because

$$q^p - 1 = (q-1)(1 + q + \cdots + q^{p-1}) = (q-1)p_q. \tag{3.43}$$

Our proof will also require a lemma about the effects of cycling through a tiling, as we did with the columns in the previous proof.

Lemma 3.1. *Let p be prime and T_0 be a tiling of a strip of length p with at least one red tile and at least one blue tile. Following the procedure outlined in the proof of Identity 17, create p distinct tilings $(T_0, T_1, \ldots, T_{p-1})$ where T_i is constructed by removing the front i tiles from T_0 and placing them at the back. (See Figure 6 for an example of this shifting procedure applied to a tiling of length 5.) Then,*

$$\sum_{i=0}^{p-1} q^{w_{T_i}} \equiv 0 \ (mod\ p_q). \tag{3.44}$$

Proof. We first note that it is enough to show that the exponents $w_{T_0}, \ldots, w_{T_{p-1}}$ are all distinct $\pmod{p}$. If each of the exponents are different $\pmod{p}$, then by applying (3.42), the sum of the $q^{w_{T_i}}$'s will be equivalent to $1 + q + q^2 + \cdots + q^{p-1} \pmod{p_q}$, giving the desired result.

Now, we consider the effect on a tiling of moving a single tile from the front to the back. Assume our tiling has k blue squares and $p - k$ red squares, where $0 < k < p$ because our tiling has both blue and red squares. If the front tile is blue, then moving it to the back will change the weight of the tiling by a factor of q^{p-k}, because we now have an additional blue

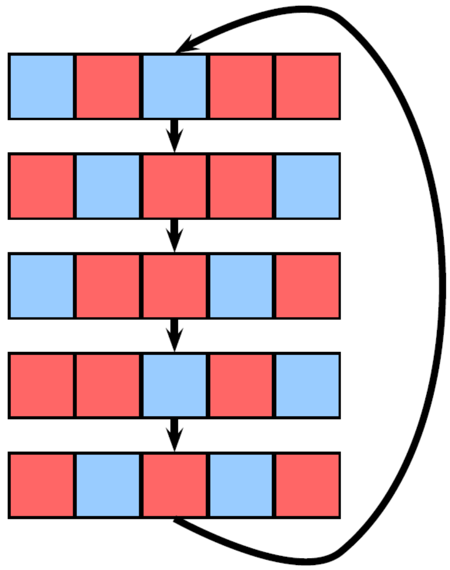

FIGURE 6. The cycling procedure applied to a tiling of length 5.

square counting all the reds. If, on the other hand, the front tile is red, moving it to the back will change the weight of the tiling by a factor of q^{-k}, because the red tile was previously counted by all the blues and is now counted by none of them. However, since $q^p \equiv 1 (\text{mod } p_q)$, these actions have equivalent effects on the weight of the tiling (mod p_q). In either case, $w_{T_1} \equiv w_{T_0} - k \pmod p$. Likewise, $w_{T_2} \equiv w_{T_0} - 2k \pmod p$, and in general,

$$w_{T_i} \equiv w_{T_0} - ik \pmod p. \tag{3.45}$$

Since p is prime and $0 < k < p$, this gives $w_{T_0}, \ldots, w_{T_{p-1}}$ all distinct (mod p), as claimed. ☐

With this lemma in place, we now provide a proof of the q-Lucas' Theorem.

Proof of the q-Lucas' Theorem. Consider the question of the weight of the collection of tilings (mod p_q) of a board of dimensions $p \times n$ and a strip of dimensions $a \times 1$ using $pk + b$ blue unit squares and $p(n-k) + (a-b)$ red unit squares. Note that to assign a weight to such a board/strip combination, we must provide an order for all $pn + a$ locations. We will say that the first location is the upper left corner of the board. From there, we proceed down the first column, then down the second, and so on across the board. Finally, we proceed across the strip so that the last tile is at the right end of the strip, as illustrated in Figure 7.

Despite the unusual layout, one option is to determine the sum of the weights of the tilings in the usual way. Because we are tiling $pn + a$ locations using $pk + b$ blue squares, the sum of the weights is

$$\binom{pn+a}{pk+b}_q.$$

On the other hand, we can proceed as in the proof of the standard Lucas' Theorem, collecting tilings into groups of size p in such a way that the weight of each collection is a multiple of q_p. As in the standard proof, look for the first column in which tiles are not all the same color and perform the same shifting procedure to create p different tilings. By Lemma 3.1, the sum of the weights of those cycled columns will be a multiple of p_q. However, we are concerned with the weights of the complete board/strip tilings and not just the cycled columns. Because

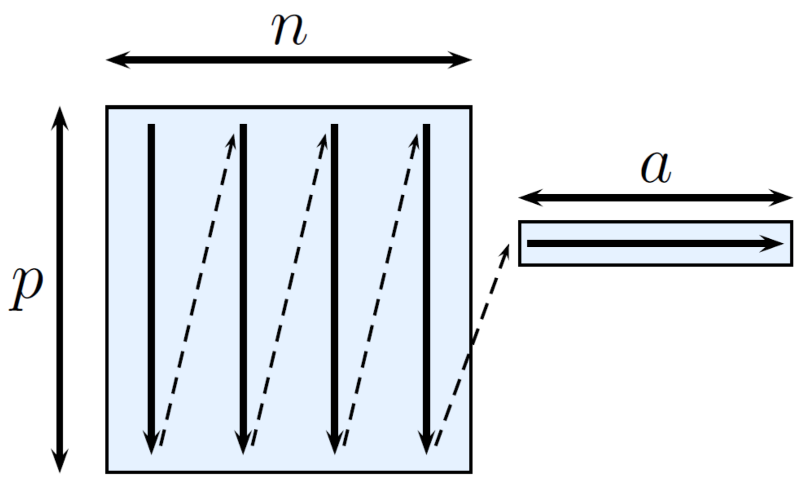

Figure 7. An illustration of the order in which tiles should be "read" off the combined board and strip.

shifting the tiles within a column does not change the interactions between that column and the rest of the tiling, the sum of the weights of the any such collection of board/strip tilings is also a multiple of p_q. Hence, because we are working modulo p_q, we can disregard any tilings with any polychromatic columns and consider only those tilings in which every column is monochromatic.

To calculate the sum of the weights of these tilings, we separate the contributions from the board and the strip. First we consider the weight of just the board, then the interaction between the board and strip, and finally the weight of just the strip.

In the collection of tilings with monochromatic columns, each board must consist of k columns of blue tiles and $n-k$ columns of red tiles. Because each column has height p, we see that every blue column contributes weight q^{p^2} for each red column before it. However, because we are working modulo p_q, we know $q^p \equiv 1$ and therefore, $q^{p^2} \equiv 1$ as well. Hence, the weight of the board will always be 1 (mod p_q).

What about the interaction between the board and the strip? The blue squares in the strip will gain some weight because the red squares in the board are "behind" them. How much weight? Each of the b blue squares in the strip counts each of the $p(n-k)$ red squares in the board, for a total extra weight of $q^{bp(n-k)}$. However, modulo p_q, we have $q^{bp(n-k)} \equiv 1$, so neither the board nor the board/strip interaction ever contributes any weight (mod p_q).

Finally, fix some particular board with k blue columns and $n-k$ red columns. For any such board, the sum of the weights of the strips will be $\binom{a}{b}_q$, and hence, the sum of the weights of the board/strip combinations will be $\binom{a}{b}_q \pmod{p_q}$. Because there are $\binom{n}{k}$ different possible boards, the total weight of this collection of tilings with monochromatic columns is

$$\binom{n}{k}\binom{a}{b}_q \pmod{p_q}.$$

It follows that

$$\binom{pn+a}{pk+b}_q \equiv \binom{n}{k}\binom{a}{b}_q \pmod{p_q},$$

as claimed. □

4. Generalizations to Tilings with More Than Two Colors

4.1. **The q-multinomial Coefficient.** The standard lattice path interpretation of the q-binomial coefficients has a natural extension to the q-multinomial coefficients. We might interpret the standard multinomial coefficient $\binom{n}{n_1,n_2,n_3}$ to be the number of three-dimensional lattice paths from $(0,0,0)$ to (n_1,n_2,n_3). The q-multinomial coefficient $\binom{n}{n_1,n_2,n_3}_q$ can then be taken to be the sum of the weights of those lattice paths, where the weight of a three-dimensional lattice path is the sum of the weights of its two-dimensional projections onto the xy, xz, and yz planes. Visualization of such lattice paths quickly becomes intractable, even in three dimensions.

In our tiling interpretation, the use of additional dimensions can be replaced by the use of additional colors. Here, we will define a weighting scheme for such a tiling and then, show that it matches the natural algebraic definition for the q-multinomial coefficient.

Assume we have a total of c colors ranked from color 1, which is the "reddest," to color c, which is the "bluest." Then the standard multinomial coefficient $\binom{n}{n_1,\dots,n_c}$ counts the number of tilings of a board of length n using a total of n_i tilings of color i for each $i \in \{1,\dots,c\}$. As with the q-binomial coefficient, we provide a definition for the weight w_T of any such board T, and take the q-multinomial coefficient to be the sum of q^{w_T} over the set of all such boards.

We define our weighting scheme by saying that a tile counts all the tiles to its left that are of lower rank (i.e., "redder") than itself. Note that this is a generalization of the q-binomial coefficient. In our combinatorial definition of the q-binomial coefficient, we specified that red tiles receive no weight (there are never any tiles that are "redder" than red) and blue tiles count the red tiles to their left. Thus, in the case where $c = 2$, the tilings receive the same weight under the q-binomial and the q-multinomial weighting scheme.

Now, suppose we want to create a tiling of length n using n_i tiles of color i for each $i \in \{1,\dots,c\}$, where $\sum_{i=1}^{c} n_i = n$. We can start by placing the bluest tiles and working our way down the ranks to the reddest tiles. It is convenient here to think of the polynomial $\binom{n}{n_c}_q$ as being a sum over the $\binom{n}{n_c}$ tilings in the collection $\mathcal{T}_{n,n_c}$. Then, the weights of the n_c bluest tiles will be one of the $\binom{n}{n_c}$ terms in the sum $\binom{n}{n_c}_q$. Once these n_c tiles are placed and their weights determined, we can ignore them entirely, because they will not be contributing any additional weight to the lower-ranked tiles. We can then proceed to the next bluest tiles, the n_{c-1} tiles of rank $c-1$. We place these n_{c-1} tiles on the remaining $n-n_c$ positions. The resulting weight of just these n_{c-1} tiles is, as before, one of the terms in the sum $\binom{n-n_c}{n_{c-1}}_q$. Continuing in this fashion, we find that the total sum of the weights of the tilings is

$$
\begin{aligned}
&\binom{n}{n_c}_q \binom{n-n_c}{n_{c-1}}_q \binom{n-n_c-n_{c-1}}{n_{c-2}}_q \cdots \binom{n_2+n_1}{n_2}_q \\
=&\frac{[n]_q!}{[n_c]_q![n-n_c]_q!}\frac{[n-n_c]_q!}{[n_{c-1}]_q![n-n_c-n_{c-1}]_q!}\frac{[n-n_c-n_{c-1}]_q!}{[n_{c-2}]_q![n-n_c-n_{c-1}-n_{c-2}]_q!}\cdots\frac{[n_2+n_1]_q!}{[n_2]_q![n_1]_q!} \\
=&\frac{[n]_q!}{[n_c]_q![n_{c-1}]_q![n_{c-2}]_q!\cdots[n_1]_q!}.
\end{aligned}
$$

This final formula is precisely the algebraic definition of the q-multinomial coefficient. Thus, our combinatorial definition, in which the q-multinomial coefficient gives the weighted sum of multicolored tilings of a board of length n, can be used to provide proofs of q-multinomial identities. Furthermore, the weighting scheme for tilings with more than two colors proves useful in proving some generalizations of standard binomial identities.

4.2. **An Identity Requiring More Than Two Colors.** The following standard binomial identity is simple to prove using a tiling interpretation of the standard binomial coefficient.

$$\binom{n}{m}\binom{m}{p} = \binom{n}{p}\binom{n-p}{m-p}. \tag{4.1}$$

The most intuitive proof under the tiling interpretation of the binomial coefficient is to note that both sides of the equation are straightforward answers to the question, "How many ways are there to tile a board of length n using $n-m$ red squares, $m-p$ blue squares, and p ultra-blue[1] squares?" This proof generalizes easily to a q-analogue using the weighting scheme for three-colored tilings that we developed for the q-multinomial coefficient.

Identity 19. *For $0 \le p \le m \le n$,*

$$\binom{n}{m}_q\binom{m}{p}_q = \binom{n}{p}_q\binom{n-p}{m-p}_q. \tag{4.2}$$

Proof. Analogously with the proof of the standard binomial identity, we ask for the sum of the weights of the tilings of a board of length n using $n-m$ red squares, $m-p$ blue squares, and p ultra-blue squares. We use the weighting scheme for tilings developed for the multinomial coefficient, namely that each red square gets weight 1, a blue square preceded by r red squares gets weight q^r, and an ultra-blue square preceded by a combined total of s red or blue squares gets weight q^s.

One way to construct a tiling in this collection is by first choosing the location of the m blue and ultra-blue squares. The weight contribution of the $n-m$ red squares to the m blue and ultra-blue squares will be one of the $\binom{n}{m}$ terms in the sum $\binom{n}{m}_q$. We can then ignore the contribution of the red squares and focus on the remaining m blue and ultra-blue squares. The weight contribution of the $m-p$ blue squares to the p ultra-blue squares is then one of the terms in $\binom{m}{p}_q$. Hence, the total weight of the tilings in this collection is given by $\binom{n}{m}_q\binom{m}{p}_q$.

A second way to construct a tiling in this collection is by first choosing the location of the p ultra-blue squares. The weight contribution of these p ultra-blue squares to the tiling as a whole will be one of the terms in the sum $\binom{n}{p}_q$. We can then ignore the ultra-blue squares and focus on the location and weight contribution of the remaining $m-p$ blue and $n-m$ red squares. This weight contribution is one of the terms in $\binom{n-p}{m-p}_q$. Thus, the total weight of the tilings in our collection is given by $\binom{n}{p}_q\binom{n-p}{m-p}_q$.

Thus, because both provide answers to the same combinatorial question, it follows that

$$\binom{n}{m}_q\binom{m}{p}_q = \binom{n}{p}_q\binom{n-p}{m-p}_q,$$

as claimed. □

[1]For our purposes, think of "ultra-blue" as a color that is so mind-bogglingly blue that even blue itself looks red in comparison.

4.3. **An Alternating-Sum Multinomial Identity.** The following alternating-sum identity, taken from Benjamin and Quinn [3], generalizes in an interesting way. The generalization to a q-identity requires the introduction of q-multinomial coefficients, even though the original identity contains only standard binomial coefficients.

We present first the standard identity and its proof, followed by a q-generalization.

Identity 20. *For $n \geq 0$,*

$$\sum_{k=0}^{n} \binom{n}{k} (n-k)^n (-1)^k = n! \tag{4.3}$$

Proof. Consider the collection of tilings of length n with the following properties. Firstly, each of the n squares in a tiling must be "light" or "dark". Secondly, every square in a tiling must point to a light square. (The light/dark notation here is intentionally different than the red/blue color distinction. In the q-generalization, tiles will still be labeled as light or dark, with weight determined by a color we will assign later, and unaffected by lightness or darkness.)

Now, let k count the number of dark squares in a tiling. Suppose we are interested in how many more tilings have even values of k than have odd values of k.

On the one hand, we can answer this question with an alternating sum. For any fixed value of k, the number of tilings in the collection that have exactly k dark squares is given by

$$\binom{n}{k} (n-k)^n.$$

(First choose the location of the k dark squares, then choose which of the $(n-k)$ light squares each square points to.) Therefore, our answer is given by the alternating sum

$$\sum_{k=0}^{n} \binom{n}{k} (n-k)^n (-1)^k.$$

On the other hand, we can proceed by pairing up tilings of opposite parity and then counting only those tilings that do not get paired up. To create these pairs, we choose the involution that finds the first square in a tiling that is not pointed to and toggles it between light and dark. The only exceptional tilings that remain unpaired by this involution are those tilings in which every square is pointed to. Because every square is pointed to, these tilings must consist of all light squares (i.e., $k = 0$), so all of the exceptions should be counted positively. Furthermore, the number of ways to create a tiling in which every square is pointed to is the same as the number of ways to order the numbers 1 through n. That is, our answer is $n!$.

Thus,

$$\sum_{k=0}^{n} \binom{n}{k} (n-k)^n (-1)^k = n!,$$

as desired. ☐

We now prove a q-analogue in which the q-multinomial coefficient makes an appearance, despite the absence of the multinomial coefficient in the original identity and its proof.

Identity 21. *For $n \geq 0$,*

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{a_i} \binom{n}{a_1, \ldots, a_{n-k}}_q (-1)^k = n_q! \tag{4.4}$$

Proof. As in the proof of the original identity, we consider the collection of tilings of length n in which each square is light or dark and every square must point to a light square. However, instead of simply counting the number of such tilings, we assign a weight to every tiling by first assigning a color to every square. We allow squares to come in any of n colors ranked from 1 (reddest) to n (bluest), in addition to being dark or light. The color assigned to a square is given by the position of the square it points to. For example, a square that points to the square on position 5 receives color number 5. Then, the weight of the tiling as a whole is given by having each tile count all tiles to its left of lower rank than itself, as in Section 4.1.

In the context of this collection, we can now ask for the difference in weight between the subcollections of tilings, where the number of dark squares has even versus odd parity.

On the one hand, we can express this difference as an alternating sum over k, the number of dark squares in the tiling. To do so, we need to know for any fixed value of k, what the total weight is of the tilings that use exactly k dark squares. First, we may choose the location of the k dark squares in any of $\binom{n}{k}$ ways. Having selected these locations, we now have $n-k$ color choices for each of the n tiles. The sum of the weights of the tilings of length n using $n-k$ ranked colors is

$$\sum_{a_i} \binom{n}{a_1, a_2, \ldots, a_{n-k}}_q,$$

regardless of our choice of the location of the dark squares (and hence, our choice of precisely *which* $n-k$ ranked colors are available.)

Therefore, one way to express the desired difference in weights is with the alternating sum

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{a_i} \binom{n}{a_1, \ldots, a_{n-k}}_q (-1)^k.$$

On the other hand, we can describe a weight-preserving, parity-reversing involution that pairs up most of the tilings and then, consider only those tilings left unpaired. As before, we can look for the first tile that is not pointed to and toggle it between light and dark. This procedure is clearly parity-reversing. Furthermore, it has no effect on the color of the tiles, and therefore, no effect on the weight of a tiling. Thus, we need only look at exceptional tilings for which this procedure is impossible to perform, namely those tilings in which every square is pointed to. Equivalently, this exceptional set is the collection of tilings comprised of only light tiles, which use exactly one square of each color 1 through n.

As before, this exceptional set consists of $n!$ tilings, but now we want to know what their combined weight is. To create such a tiling, we can choose the location of one square at a time, starting from the bluest square. If the bluest square is placed in the first position, it will get weight 1. In the second position, it will have one lower-ranked tile to its left and therefore get weight q. In general, in position i, the bluest square will get weight q^{i-1}, so the sum of the possible weights for the bluest square is $1 + q + \cdots + q^{n-1} = n_q$.

With the bluest tile placed, now choose where to place the second-bluest tile from among the remaining $n-1$ positions. In the first of the available positions, it will get weight 1. In the second, it will get weight q, and so forth. Altogether, the sum of the possible weights for the second-bluest tile is $1 + q + \cdots + q^{n-2} = (n-1)_q$.

Continuing in this fashion, we see that the sum of the possible weights for the $i+1$st bluest tile is $(n-i)_q$. Because each choice is made independently of all the choices before it, the sum of the weights of all tilings in this collection is the product

$$n_q(n-1)_q \cdots 1_q = n_q!,$$

giving the desired result. □

4.4. **Fermat's Little Theorem.** Benjamin and Quinn [3] present a simple tiling proof to Fermat's Little Theorem, which makes use of multicolored board tilings. We first provide a sketch of their proof of Fermat's Little Theorem, and then furnish a q-analogue.

Identity 22 (Fermat's Little Theorem)**.** *For any prime p and any natural number a,*

$$a^p \equiv a \ (mod\ p). \tag{4.5}$$

Proof. Consider the number of ways (mod p) to tile a board of length p using squares that come in a different colors.

On the one hand, for each of the p positions on the board, we can independently choose from a different colors. Hence, the total number of tilings is a^p.

On the other hand, because we only care about the answer modulo p, we can collect most of the tilings into groups of size p and consider only those tilings that are left ungrouped. For each tiling that is not monochromatic, we place the tiling into the set of p tilings created by repeatedly removing the front tile and moving it to the back. Because p is prime, this procedure is guaranteed to create p distinct tilings. Thus, all polychromatic tilings can be placed into groups of size p, and all that remains (modulo p) are the a monochromatic tilings.

Thus,

$$a^p \equiv a \ (\text{mod } p),$$

as desired. □

We now prove a q-analogue using similar logic. In this analogue, the possible presence of tilings with more than two colors requires us to pick a weighting scheme for such tilings. Notably, we will use a different weighting scheme than the one developed for use in the q-multinomial coefficient.

Identity 23 (A q-analogue to Fermat's Little Theorem)**.** *For any prime p and any natural number a,*

$$\sum_{k=1}^{p} \binom{p}{k}_q \sum_{j=1}^{a} (a-j)^{p-k} \equiv a \ (mod\ p_q). \tag{4.6}$$

It may not be obvious at first glance that this is an analogue to Fermat's Little Theorem because the left side of the equation does not resemble a^p. However, we can see that

$$a^p = \sum_{k=1}^{p} \binom{p}{k} \sum_{j=1}^{a} (a-j)^{p-k}$$

by observing that both a^p and $\sum_{k=1}^{p} \binom{p}{k} \sum_{j=1}^{a} (a-j)^{p-k}$ count the number of tilings of a board of length p using squares that come in a colors. To see this, number the colors from lowest rank, 1, to highest rank, a. In each tiling, find the lowest ranked color in the tiling and call this color j. Then count the number of tiles that share this color, and call this number k. Conditioning first on the value of k, and then on the value of j gives the desired formula.

Proof of the q-analogue to Fermat's Little Theorem. Consider the weight (mod p_q) of the collection of all distinct tilings of a board of length p using squares that come in a different colors.

To find the weight of this collection, we must first define a weighting scheme for tilings in which tiles are allowed to come in colors labeled 1 (the "reddest" color) through a (the "bluest" color). To determine the weight of a tiling, we first look for the tile or tiles in that tiling of lowest rank (i.e., the "reddest" tiles). We then treat all tiles of this color as red and all other tiles as blue and determine the weight of the tiling using the usual method for two-colored tilings.

To prove the desired identity, we now provide two different calculations of the weight of this collection (mod p_q).

Firstly, we break up the tilings in the collection by conditioning on the lowest rank present and the number of tiles that share the lowest rank. Let j represent the lowest ranked color present in a tiling and k represent the number of tiles in the tiling that share color j. Then for any fixed values of j and k, we can construct the tiling by making two independent choices. First, we choose the locations of the k rank-j tiles, which completely determines the weight of the tiling. Then, we may choose which of the $a-j$ higher-ranked colors each of the remaining $p-k$ tiles takes. Hence, the total weight of all tilings, which share these values of j and k, is given by

$$\binom{p}{k}_q (a-j)^{p-k}.$$

Thus, letting j and k vary, the total weight of the collection of tilings is

$$\sum_{k=1}^{p} \binom{p}{k} \sum_{j=1}^{a} (a-j)^{p-k}.$$

Secondly, we may make use of only looking for a weight *modulo* p_q. Recall from the proof of Lemma 3.1 that if we take a bichromatic tiling of length p and create p distinct tilings by successively removing the first tile and placing it at the back, then the sum of the weights of these tilings will be a multiple of p_q. Despite our tiles now coming in a colors rather than two, the same lemma holds here for any non-monochromatic tilings. (The lemma relies on our chosen weighting scheme reducing the old two-color weighting scheme for groups of tilings going through the cycling procedure. The lemma would *not* continue to hold if we chose the same weighting scheme that we used for the q-multinomial coefficient.) Hence, we can group all non-monochromatic tilings into groups of size p, where the sum of the weights of the tilings in each group is a multiple of p_q.

The only remaining tilings are the a monochromatic tilings, all of which have weight 1. Hence, the weight of the entire collection (modulo p_q) is just a.

Thus,

$$\sum_{k=1}^{p} \binom{p}{k}_q \sum_{j=1}^{a} (a-j)^{p-k} \equiv a \pmod{p_q},$$

as desired. □

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